Computational Homology in Topological Dynamics

ACAT School, Bologna, Italy
May 26, 2012

Marian Mrozek

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Weather forecasts for Galway, Ireland from Weather Underground 2

Extended Forecast

Updated: 1:00 AM IST on June 20, 2009

Wednesday Night
Chance of Rain. Scattered Clouds. Low: 12 °C . Wind ESE 14 km/h . Chance of precipitation 50% (water equivalent of 1.76 mm).

Thursday
Scattered Clouds. High: 20 °C . Wind East 18 km/h .

Thursday Night
Scattered Clouds. Low: 13 °C . Wind East 14 km/h .

Friday
Scattered Clouds. High: 21 °C . Wind East 14 km/h .

Friday Night
Clear. Low: 10 °C . Wind ESE 14 km/h . Windchill: 9 °C .

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Extended Forecast

Updated: 7:00 PM IST on June 21, 2009

Wednesday Night
Partly Cloudy. Low: 12 °C . Wind SE 14 km/h .

Thursday
Chance of Rain. Scattered Clouds. High: 21 °C . Wind ESE 18 km/h . Chance of precipitation 20% (trace amounts).

Thursday Night
Chance of Rain. Scattered Clouds. Low: 11 °C . Wind ESE 14 km/h . Chance of precipitation 20% (trace amounts).

Friday
Clear. High: 21 °C . Wind East 10 km/h .

Friday Night
Chance of Rain. Scattered Clouds. Low: 13 °C . Wind SE 10 km/h . Chance of precipitation 40% (water equivalent of 1.39 mm).

Outline 3

- **Part I**
  - **Dynamical systems**
  - **Rigorous numerics of dynamical systems**
  - **Homological invariants of dynamical systems**
  - **Computing homological invariants**
- **Part II**
  - **Homology algorithms for subsets of $\mathbb{R}^d$**
  - **Homology algorithms for maps of subsets of $\mathbb{R}^d$**
  - **Applications**

Outline 4

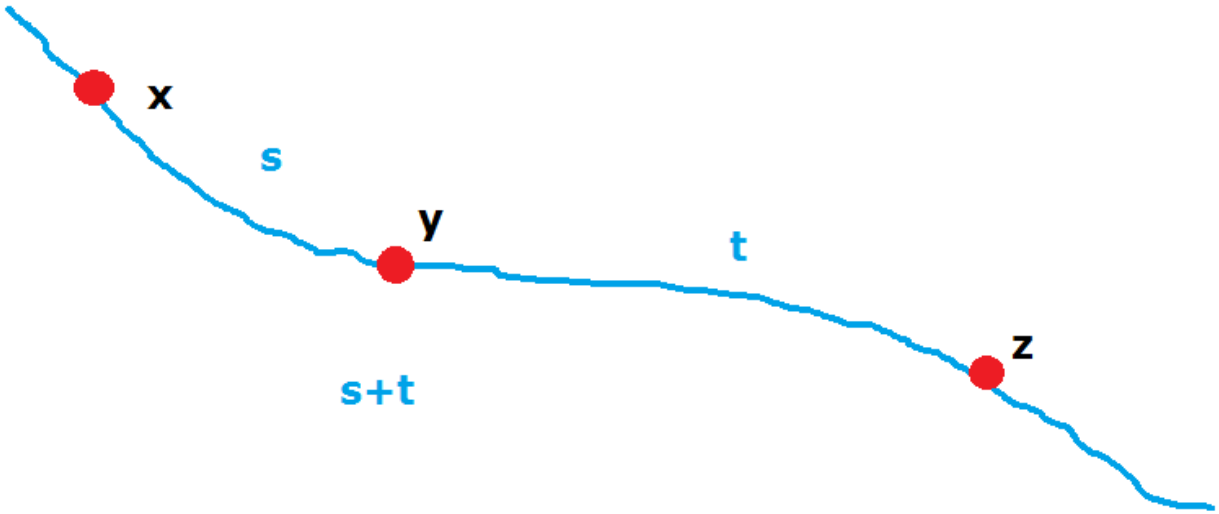
- Dynamical systems
- Rigorous numerics of dynamical systems
- Homological invariants of dynamical systems
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Dynamical systems 5

Main sources:

- Differential equations
- Iterates of maps
- Time series

Dynamical systems 6



Dynamical systems 7

- X — a compact subset of $\mathbb{R}^d$ (in general: a topological space).

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A (semi)dynamical system is a continuous map

$$\varphi : X \times T \to X$$

such that for any $x \in X$ and $s, t \in T$

$$\varphi(\varphi(x,t),s) = \varphi(x, s+t)$$
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- $f := \varphi_1$ — the generator (for discrete time only) identified with sds

Jules Henri Poincaré 8

Henri Poincaré, 1854 - 1912

Invariant sets 9

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$$\begin{aligned} \alpha(x) &:= \{\, y \in X \mid \exists t_n \to -\infty \text{ s.t. } y = \lim \varphi(x,t_n) \,\} \\ \omega(x) &:= \{\, y \in X \mid \exists t_n \to +\infty \text{ s.t. } y = \lim \varphi(x,t_n) \,\} \end{aligned}$$

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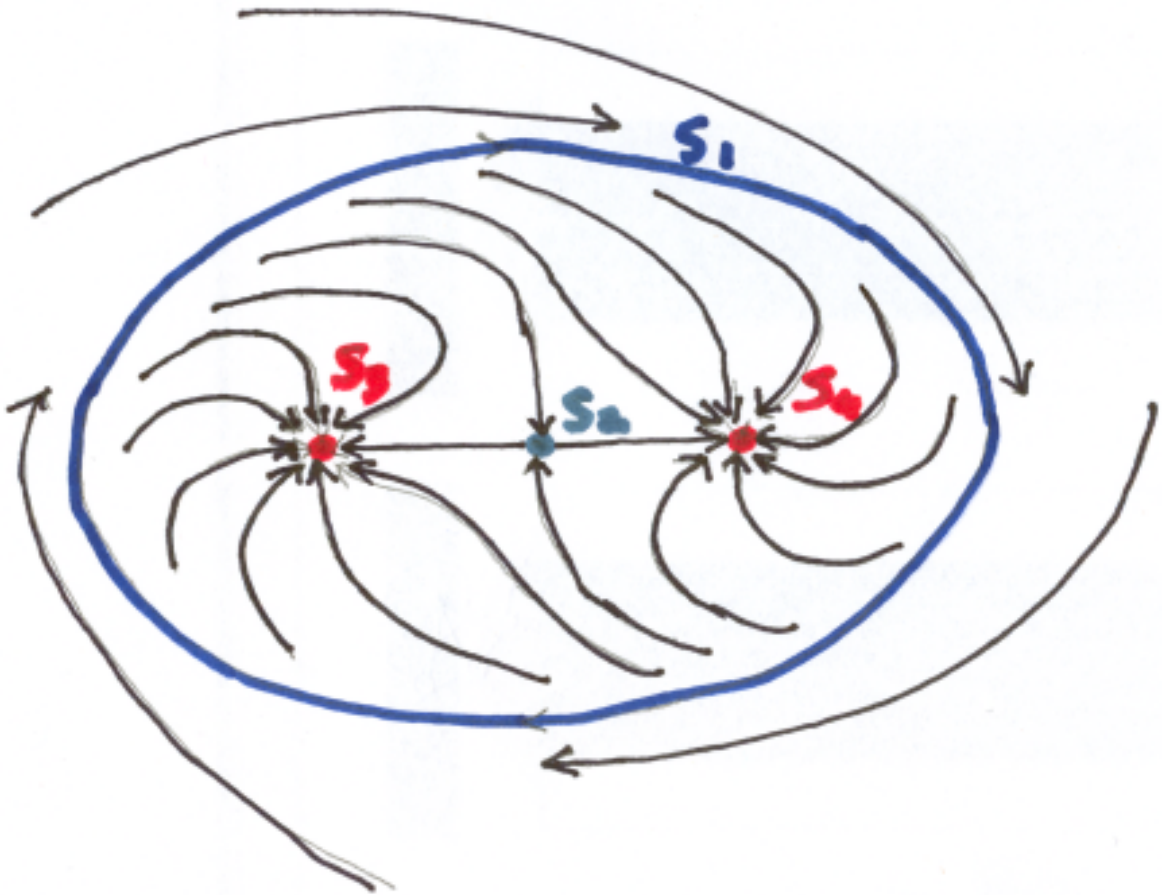
Limit sets are invariant.

Invariant sets and limit sets



Some invariant sets and limit sets.

Asymptotic dynamics

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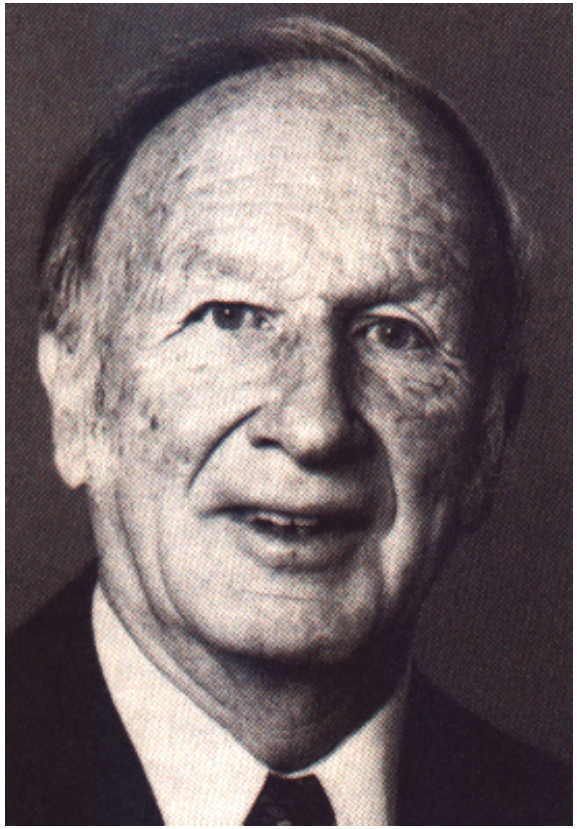
- Up to the half of the 20th century the dominating opinion was that the a limit set may be a stationary point or the trajectory of a periodic point.
- The computers significantly contributed to the realization that the asymptotic behaviour may be much more complicated (chaotic).

Edward Lorenz 12



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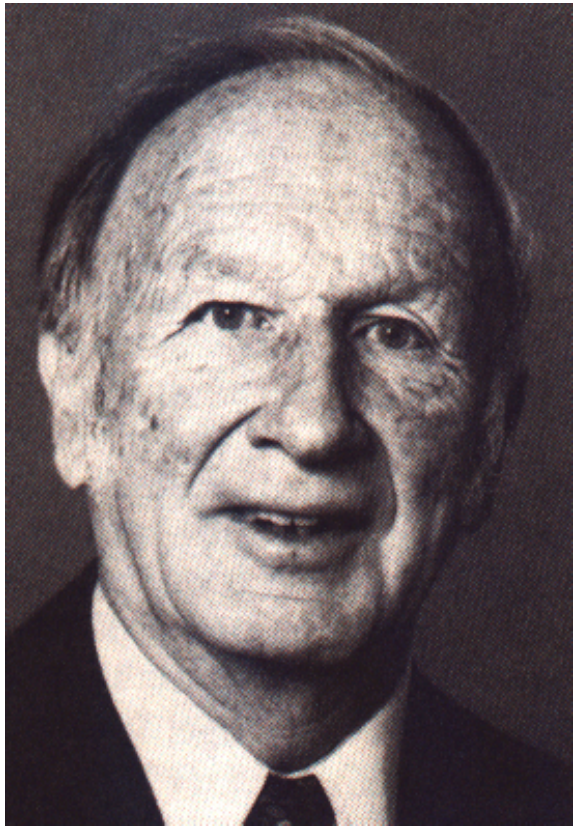

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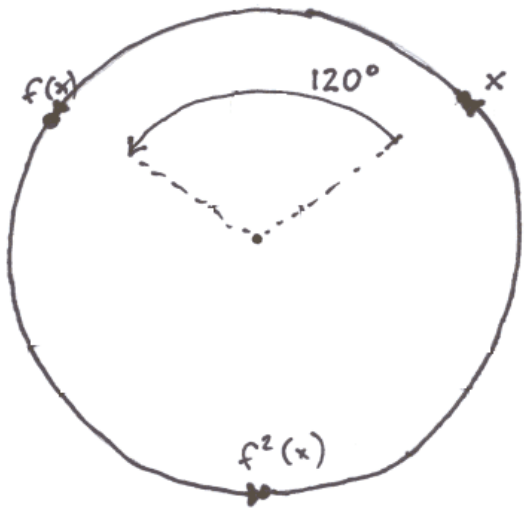
- during the 1950s became skeptical of the appropriateness of the mathematical models used in meteorology
- in 1963 published the famous paper: *Deterministic Nonperiodic Flow*
- the Lorenz equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

Main contributors to the discovery of deterministic chaos

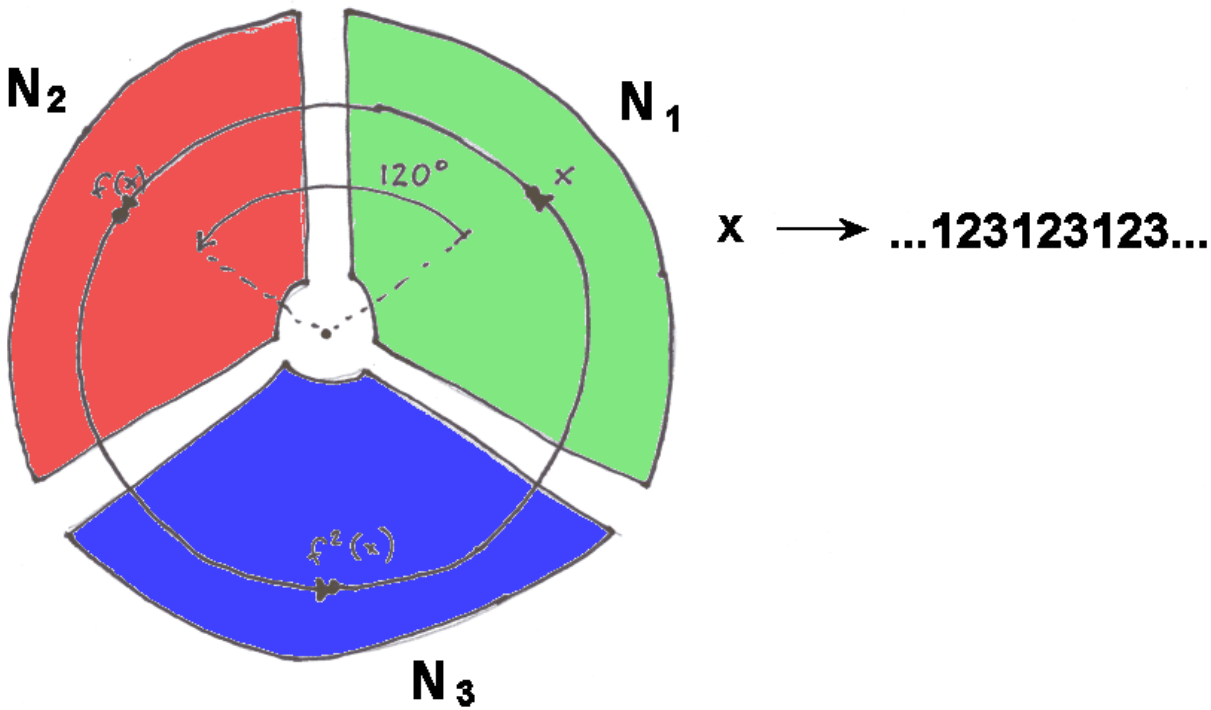
- Henri Poincaré, 1890
- Mary Cartwright and John Littlewood, 1940's
- Andrey Kolmogorov and Yakov Sinai, 1950's
- Edward Lorenz, early 1960's
- Oleksandr Sharkovsky, 1964
- Stephen Smale, 1967
- Tien-Yien Li and James A. Yorke 1975

Symbolic dynamics 14



Rotation by 120 degrees

Symbolic dynamics 15



Mapping to sequences of symbols

Shift dynamics

- Consider
$$\Sigma_k := \{0, 1, 2, \ldots k-1\}^{\mathbb{Z}}$$
as a metric space with the metric
$$d(\alpha, \beta) := \sum_{i=-\infty}^{\infty} \frac{1 - \delta_{\alpha(i),\beta(i)}}{2^{|i|}},$$
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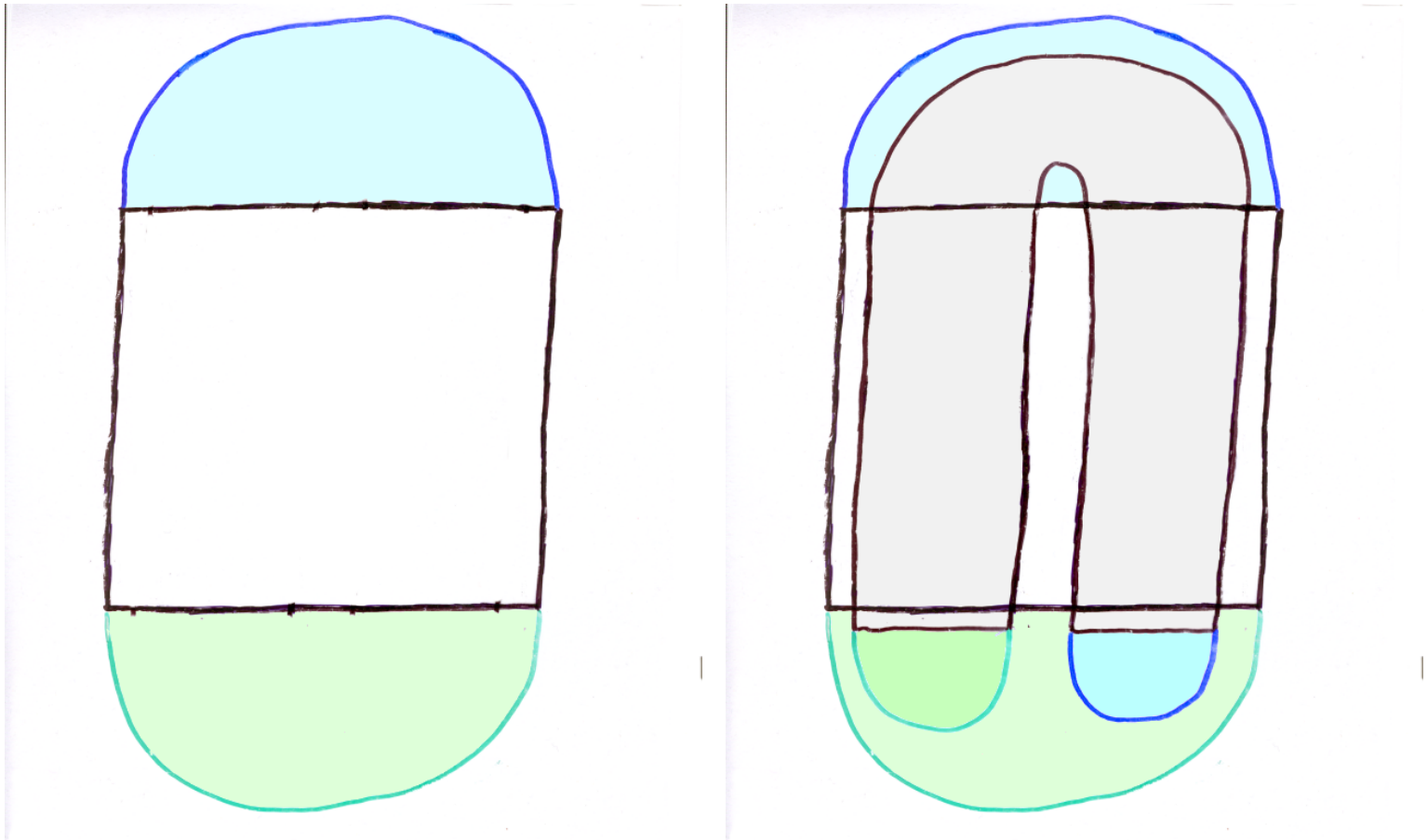
Features:

- Plenty of periodic points: Every finite sequence of symbols is in one-to-one correspondence with a periodic point of σ
- Sensitive dependence on initial conditions: trajectories diverge exponentially fast

Smale horseshoe (1967)



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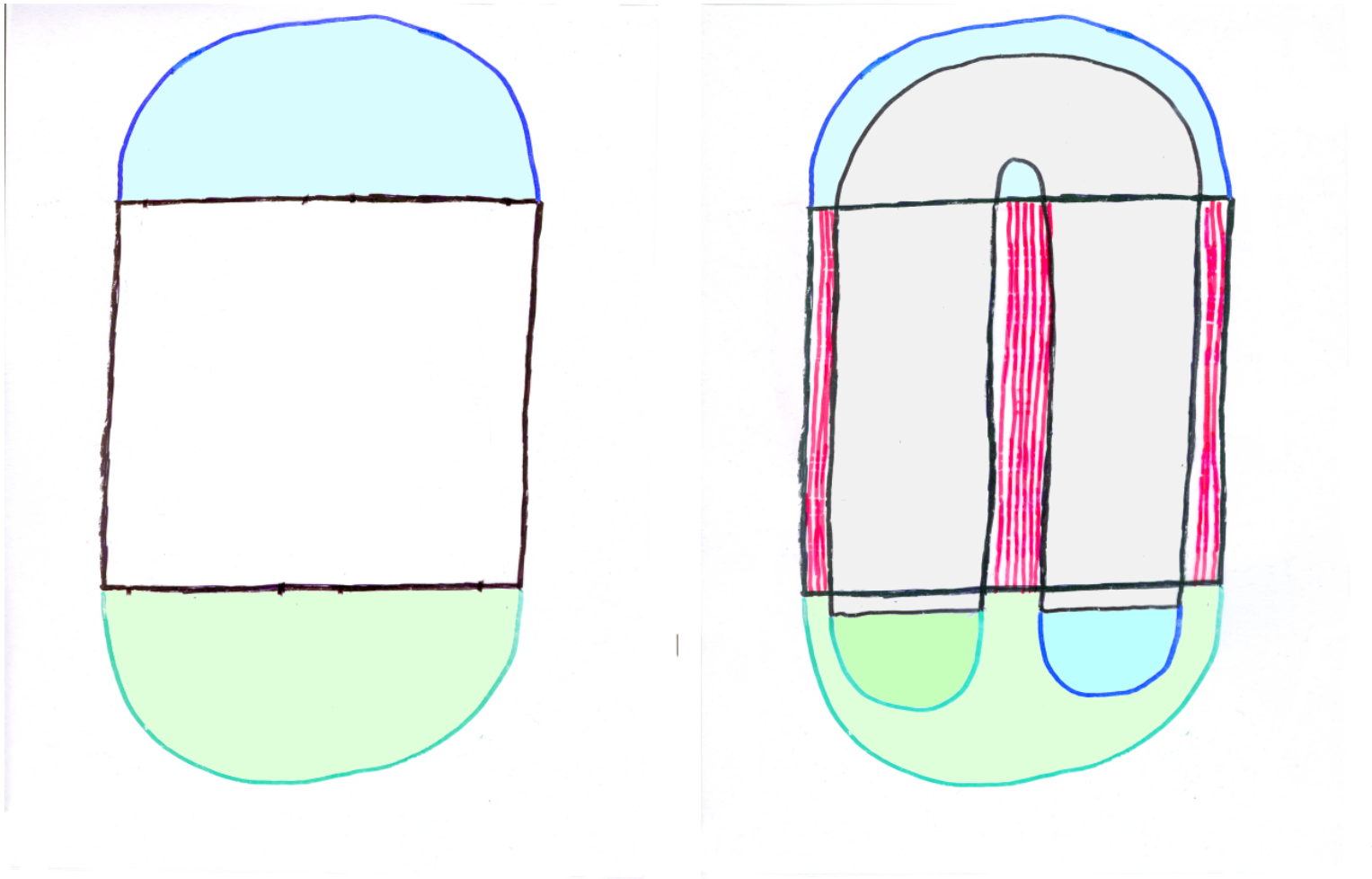

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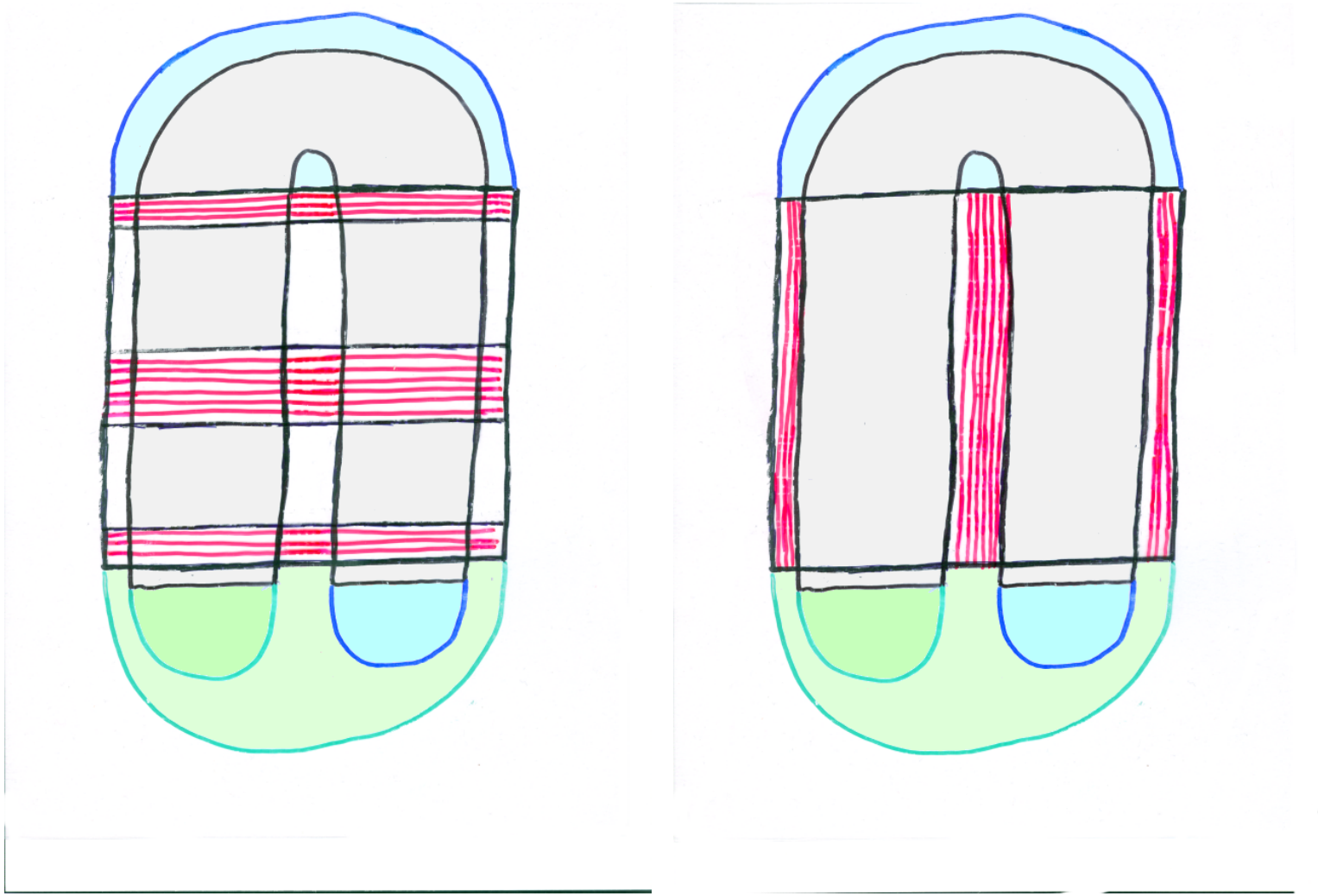

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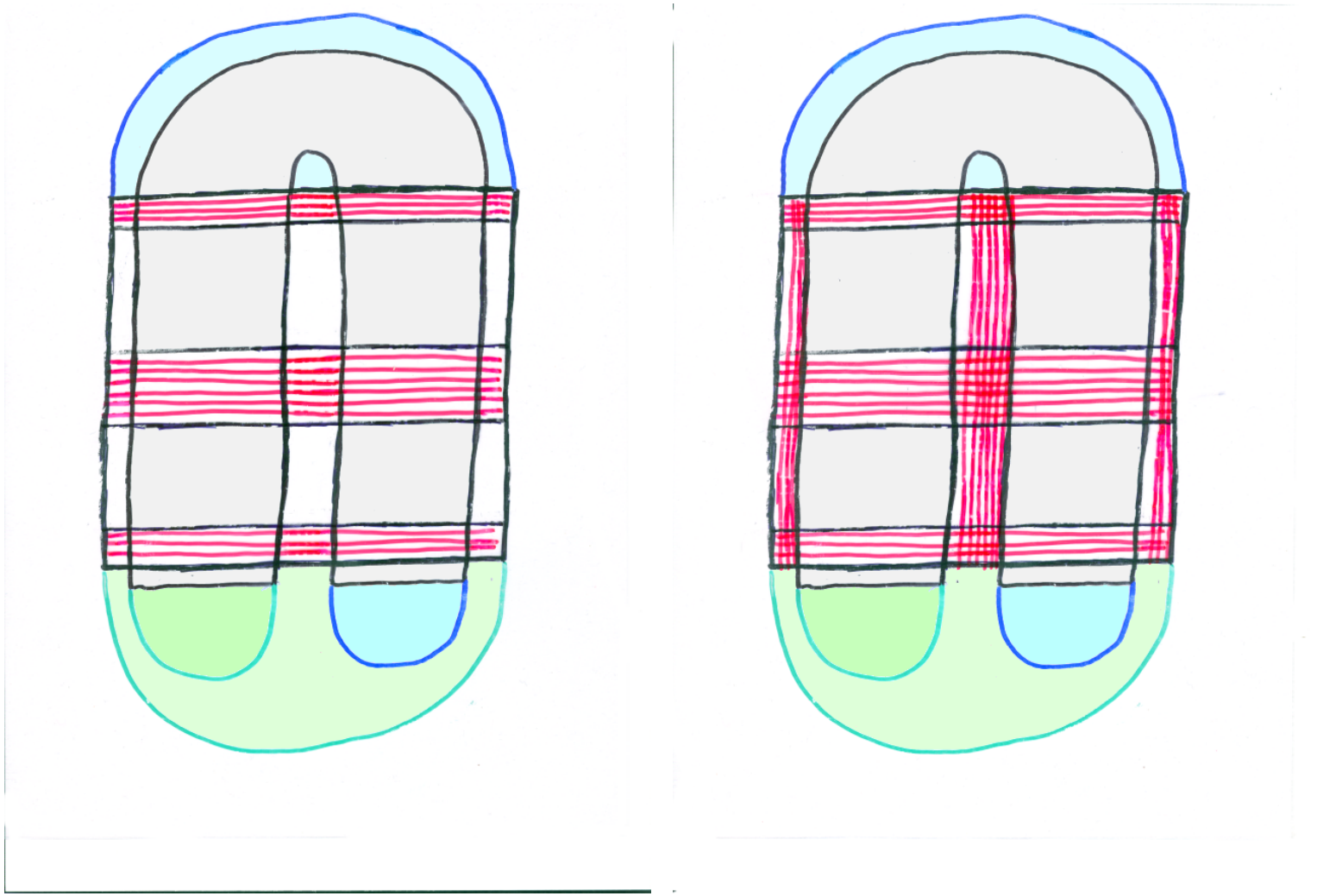

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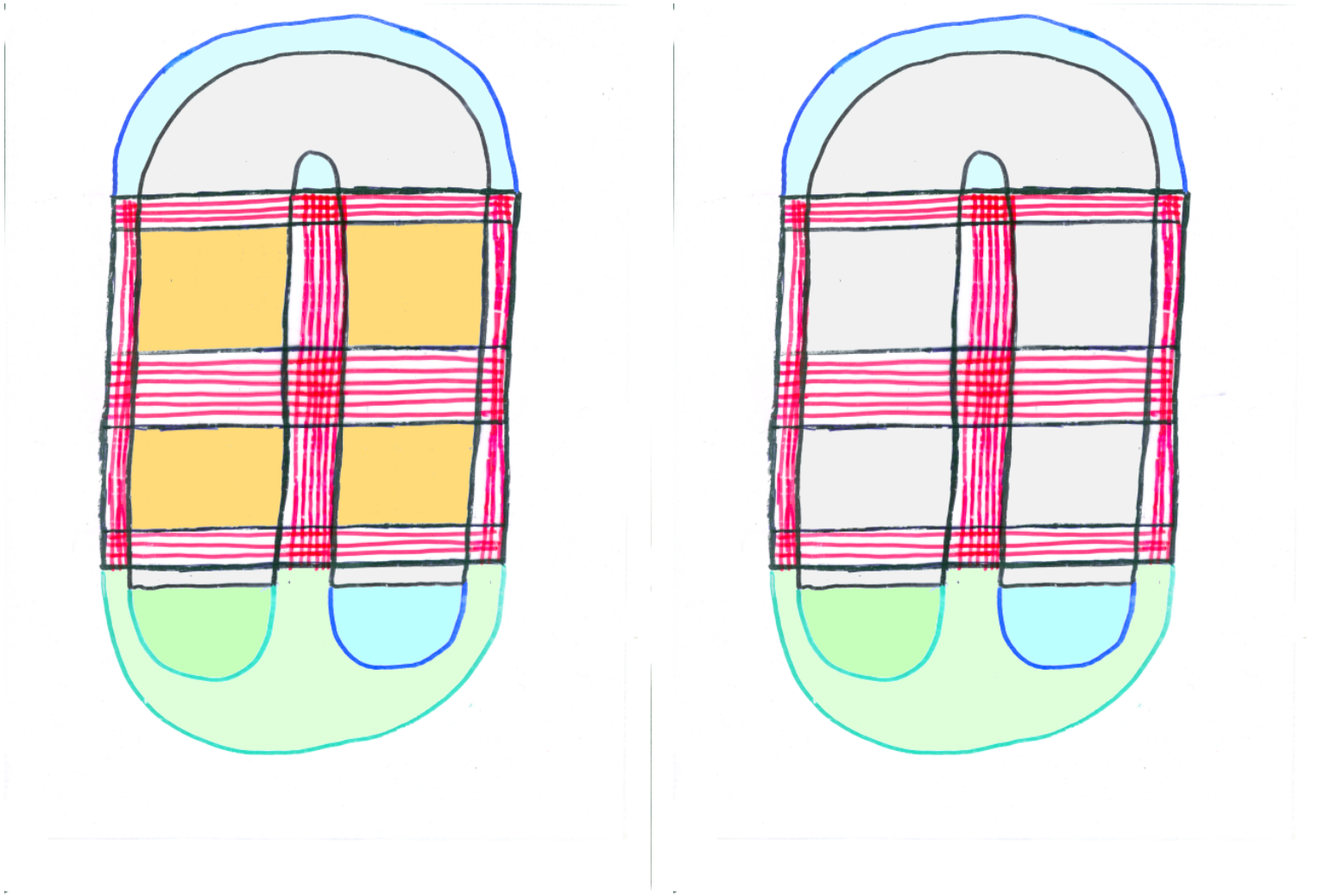

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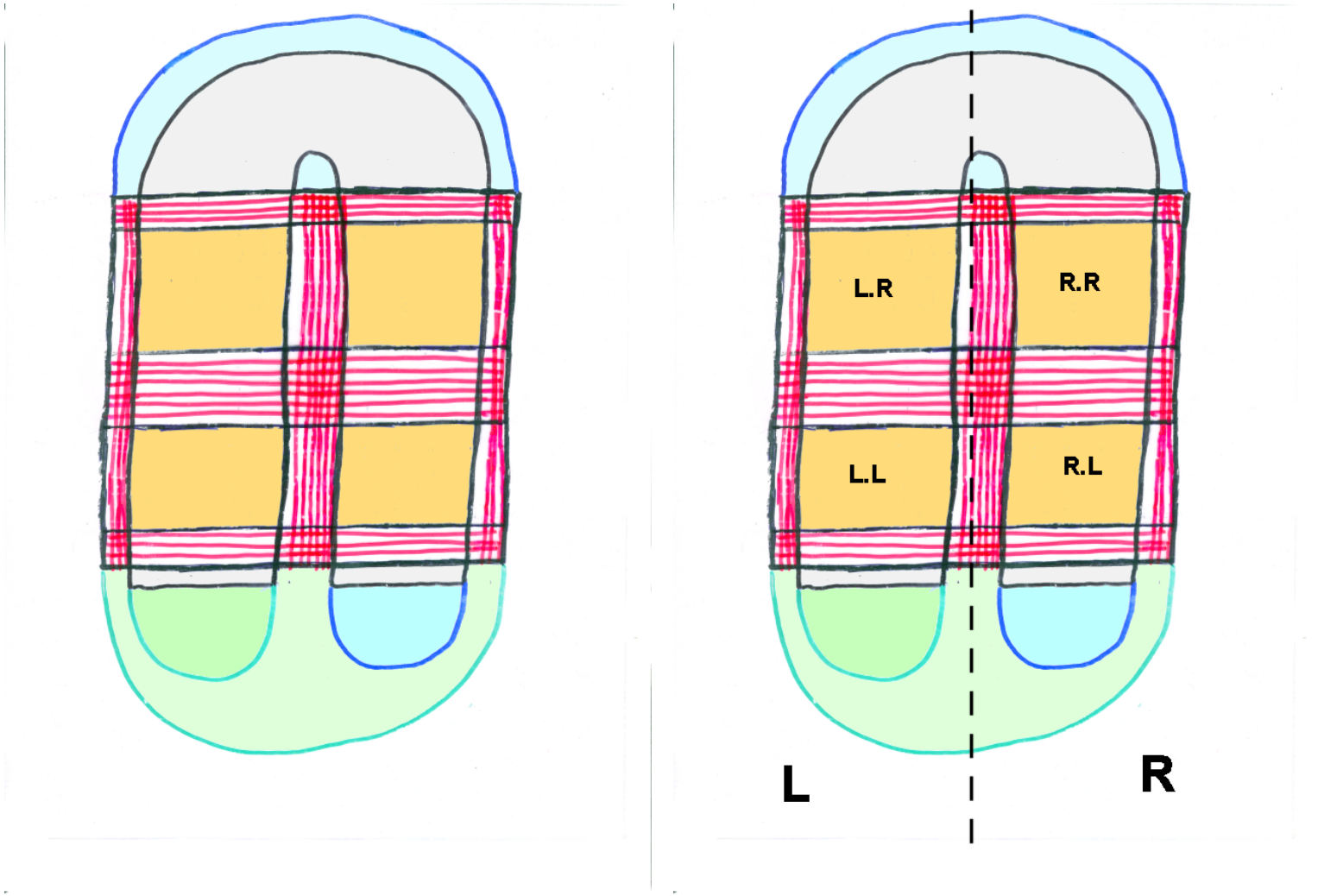

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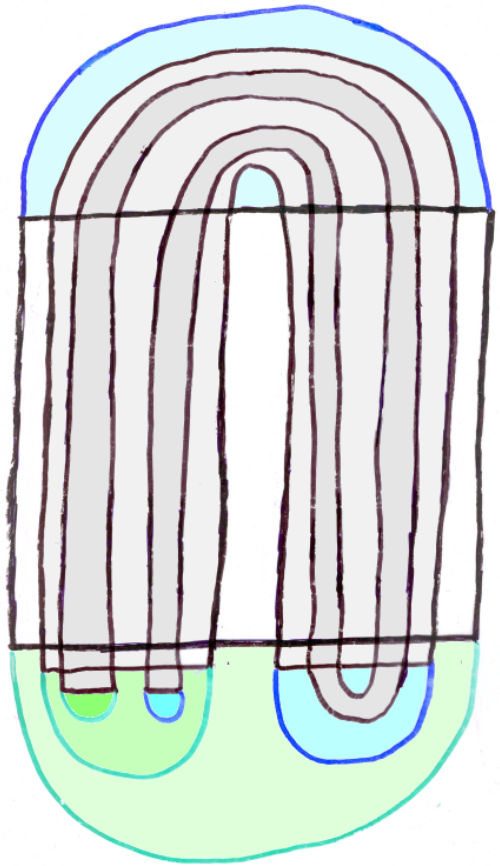

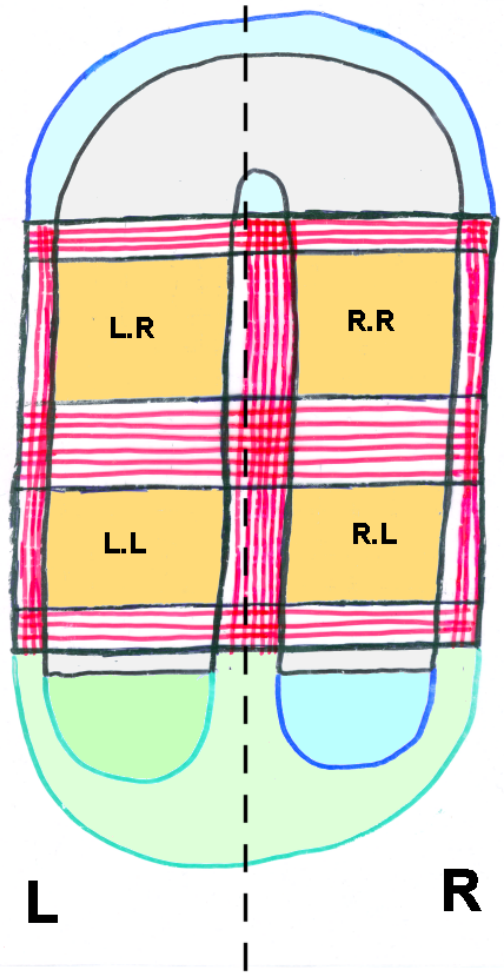

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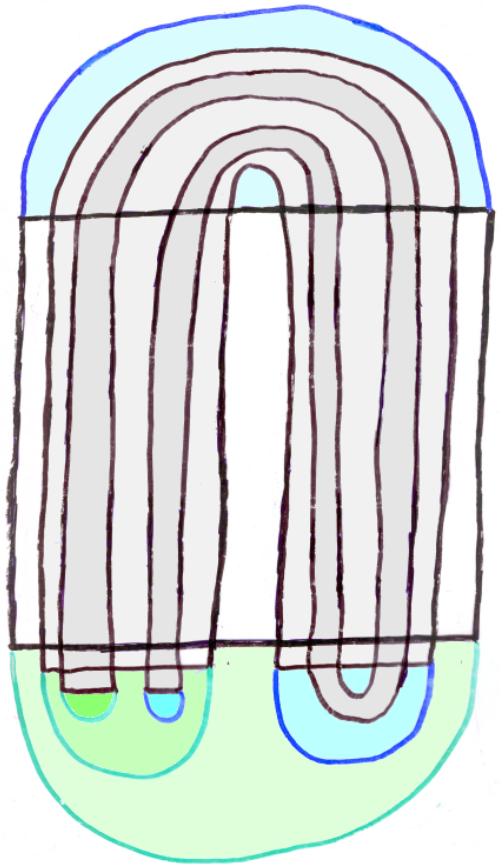
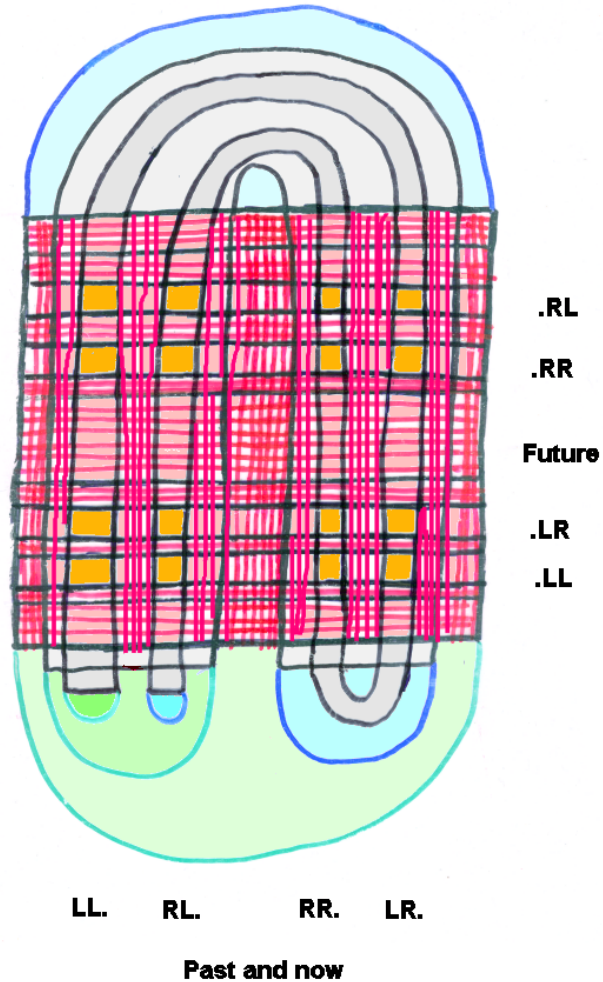
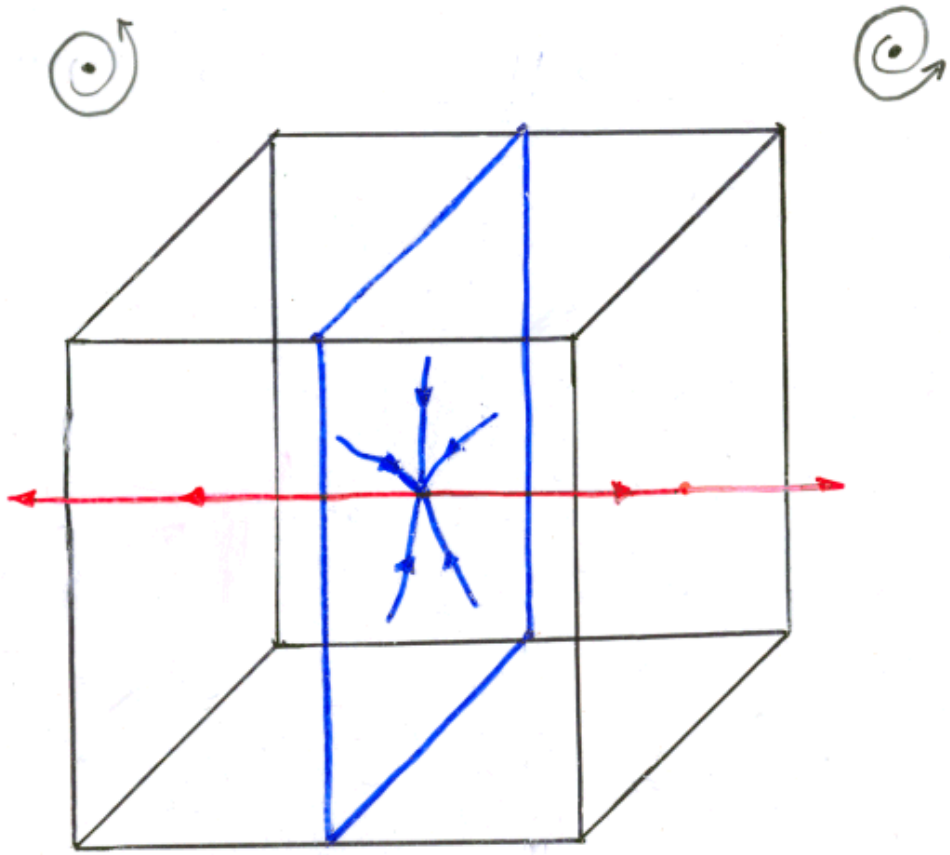




Smale horseshoe (1967)

Theorem. (Smale, 1967) Let N denote the square part of the domain of the horseshoe map h. Then there exists a homeomorphism $\rho : \mathrm{Inv}(N, h) \to \Sigma_2$ such that $\sigma\rho = \rho h$.

Lorenz equations around the origin



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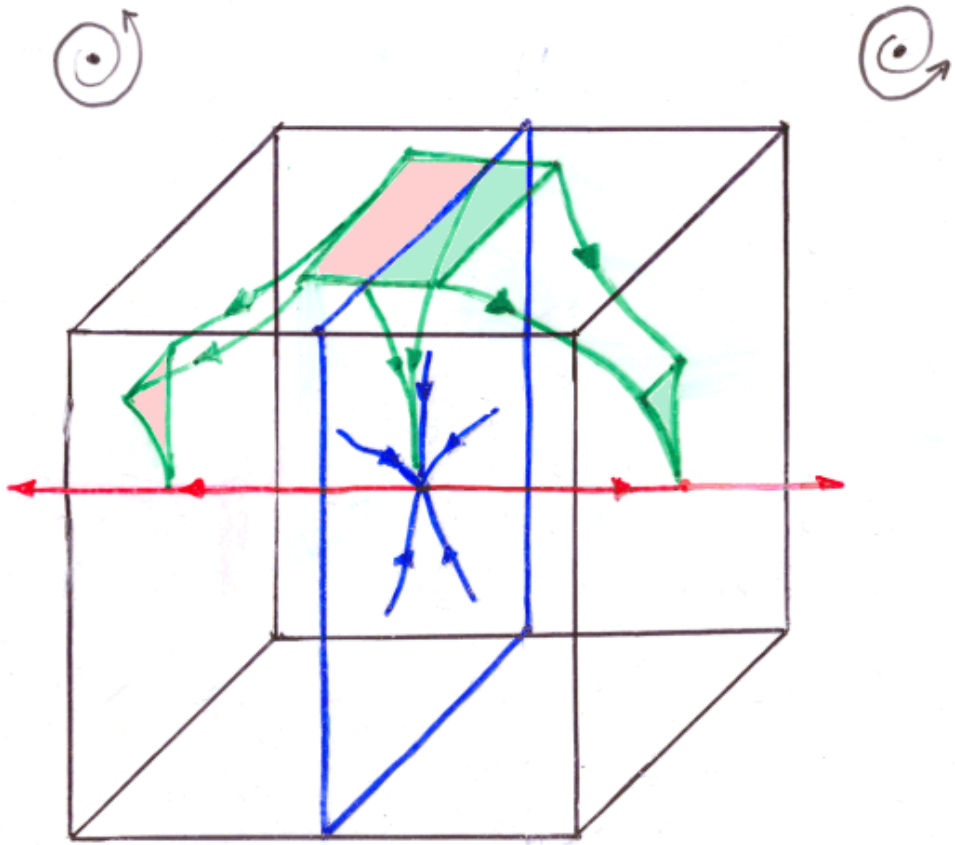

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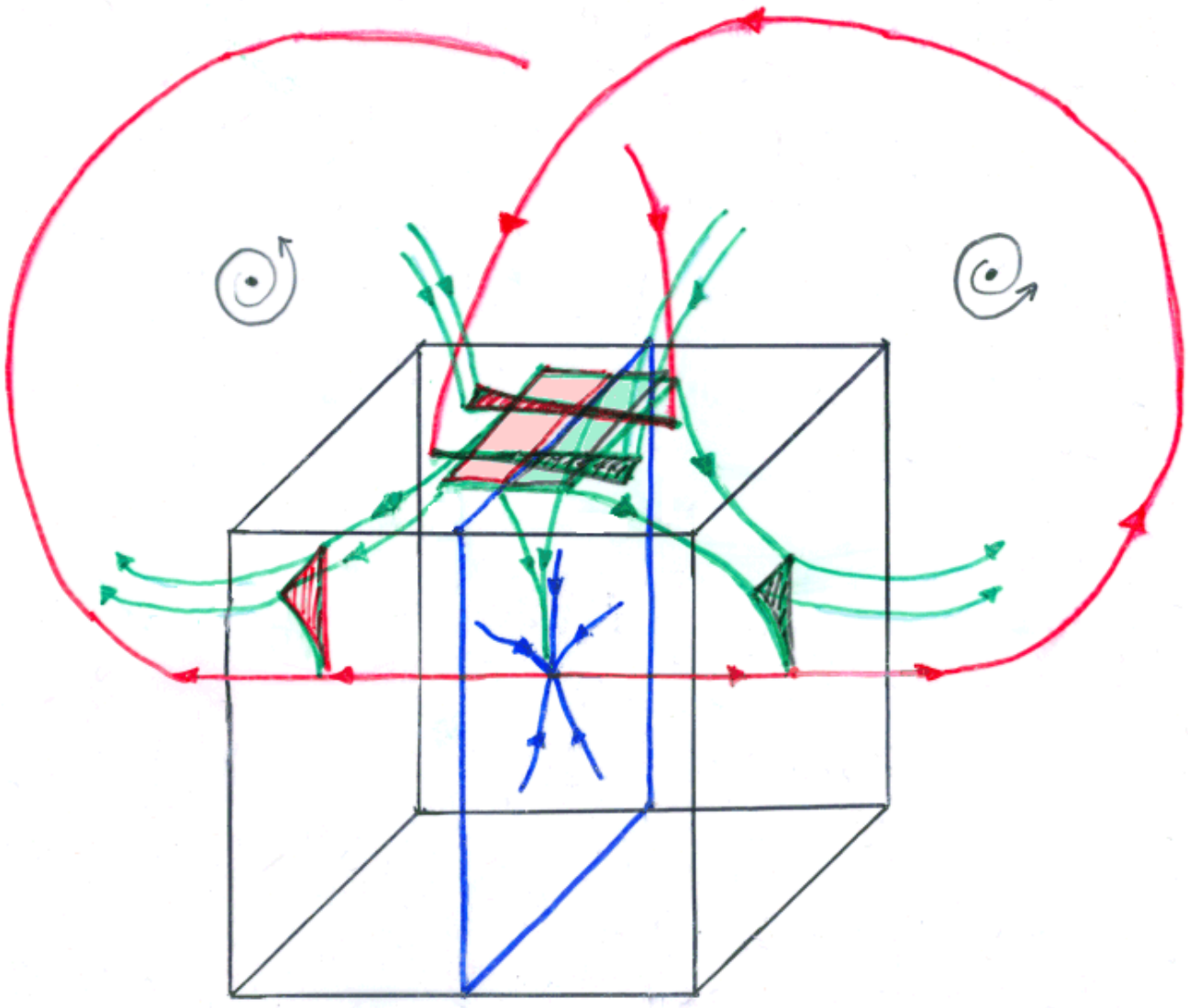
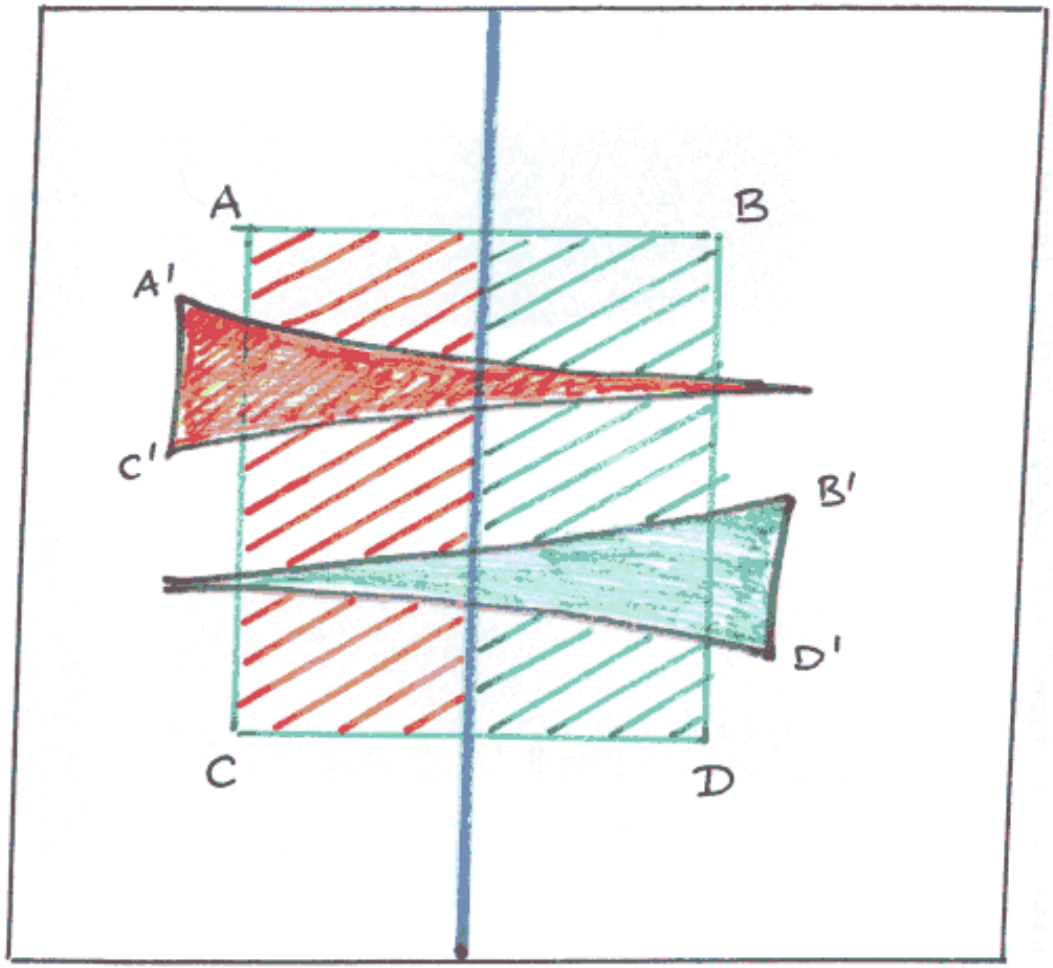


Poincaré map in the Lorenz equations



Problem 30

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- Or maybe it is only present in the numerical scheme?
- Or maybe the chaotic behaviour is only the consequence of the rounding errors?

References

- **E.N. Lorenz**, Deterministic Nonperiodic Flow, *Journal of the Atmospheric Science* (**1963**).
- **R.L. Adler, A.G. Konheim, M.H.McAndrew**, Topological entropy, *Transactions of the American Mathematical Society* (**1965**).
- **S. Smale**, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* (**1967**).
- **C. Sparrow**, The Lorenz Equations: Bifurcations, Chaos and Strange Attractors, *Springer-Verlag* (**1982**).

Outline 32

- Dynamical systems
- **Rigorous numerics of dynamical systems**
- Homological invariants of dynamical systems
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Ghost solutions

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The only periodic trajectory of this equation is the stationary point at the origin.
Consider its Euler discretization

$$\Phi_h(z) := z(1 + h(\alpha i - |z|))$$

For every $h > 0$ this discretization has invariant circles of radius

$$r_\pm := \frac{1 \pm \sqrt{1 - h^2\alpha^2}}{h}$$

Disappearing Smale's horseshoe

- The logistic equation
$$y' = y(1-y)$$
may be solved explicitly and it clearly does not exhibit chaotic behaviour
- However, Koçak and Hale (1991) prove that the two step numerical scheme
$$\Phi_{h,\lambda}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} \frac{1-\lambda}{1+\lambda}y_2 + \frac{2\lambda}{1+\lambda}y_1 + 2hy_1(1-y_2) \\ y_1 \end{pmatrix}$$
contains an invariant subset conjugate to a horseshoe for every $h > 0$.

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In both cases the failures were attributed to numerical errors.

Interval arithmetic

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Interval arithmetic 36

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Given the endpoints of I and J, one can easily construct the endpoints of $I \diamond J$.

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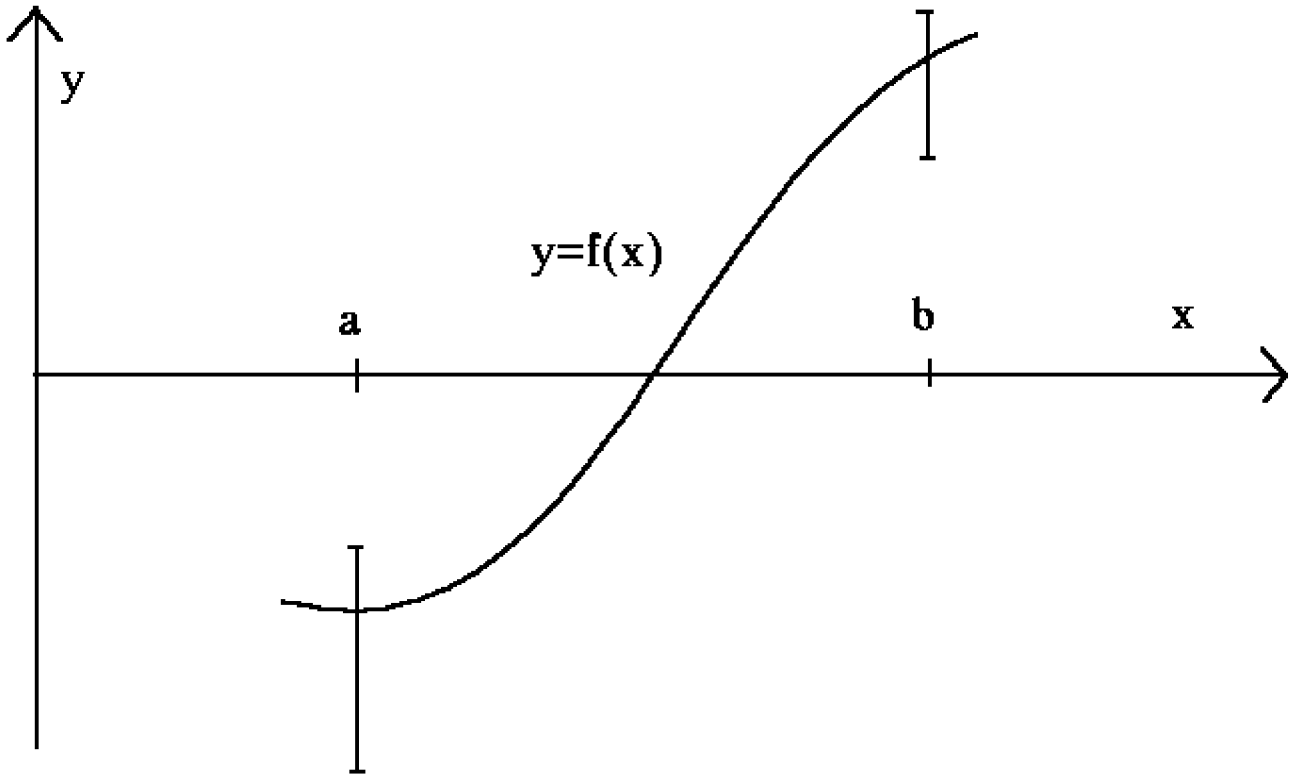
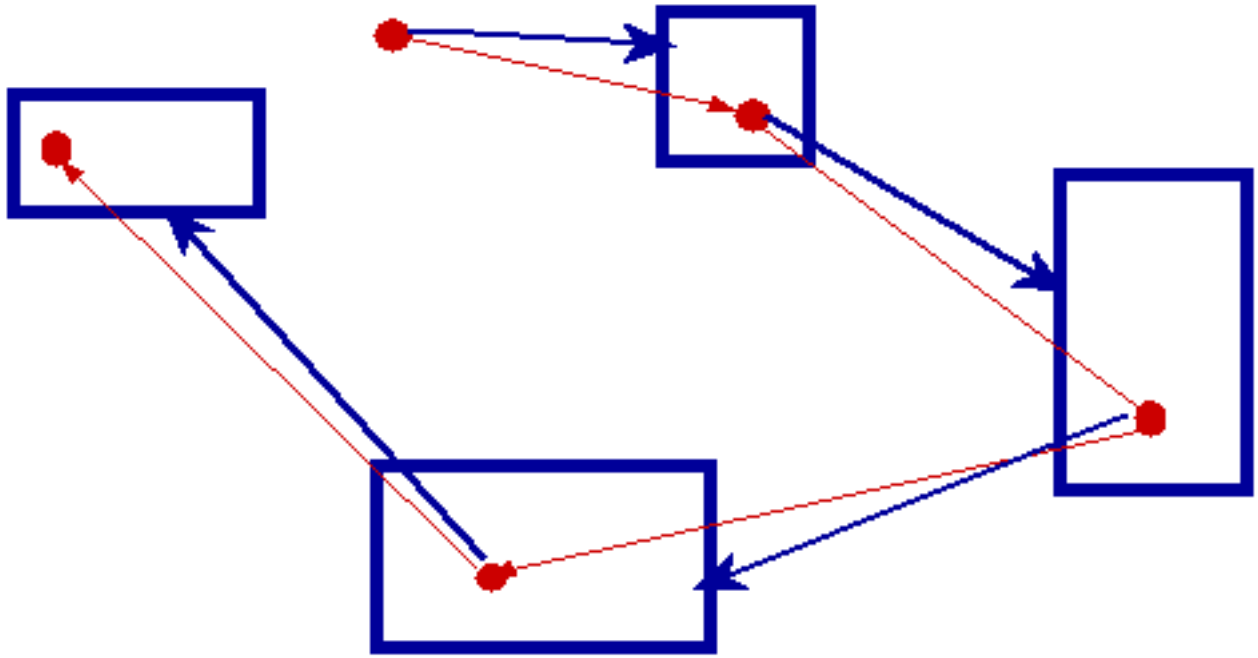


Discretization in time



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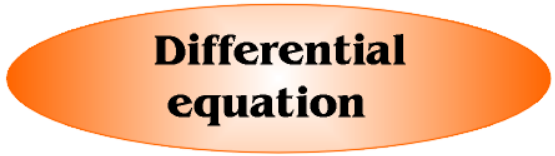


Differential equation
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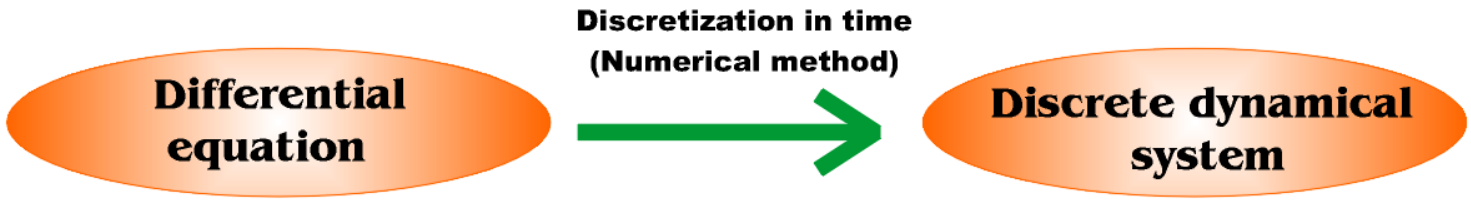
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Discrete dynamical system

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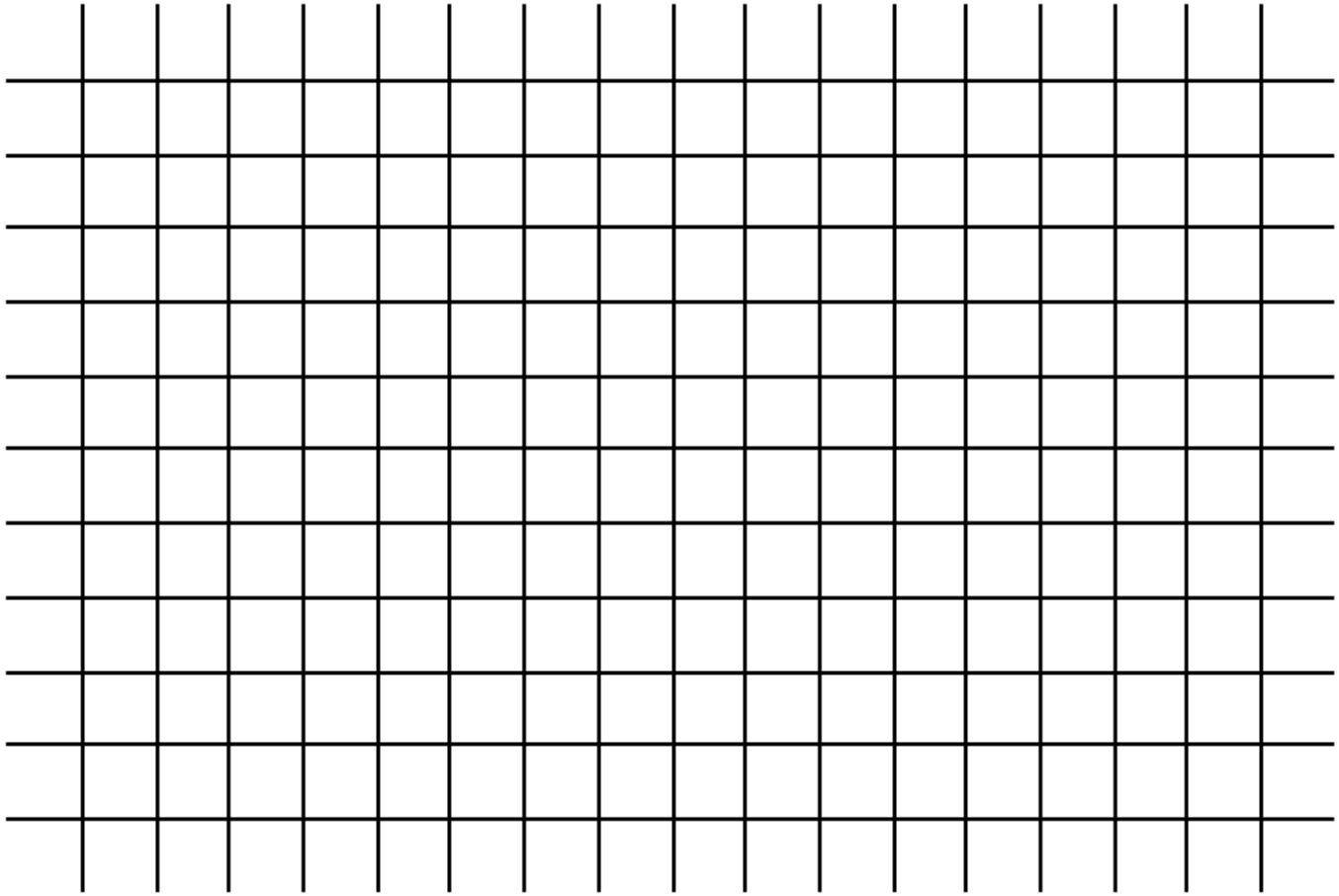

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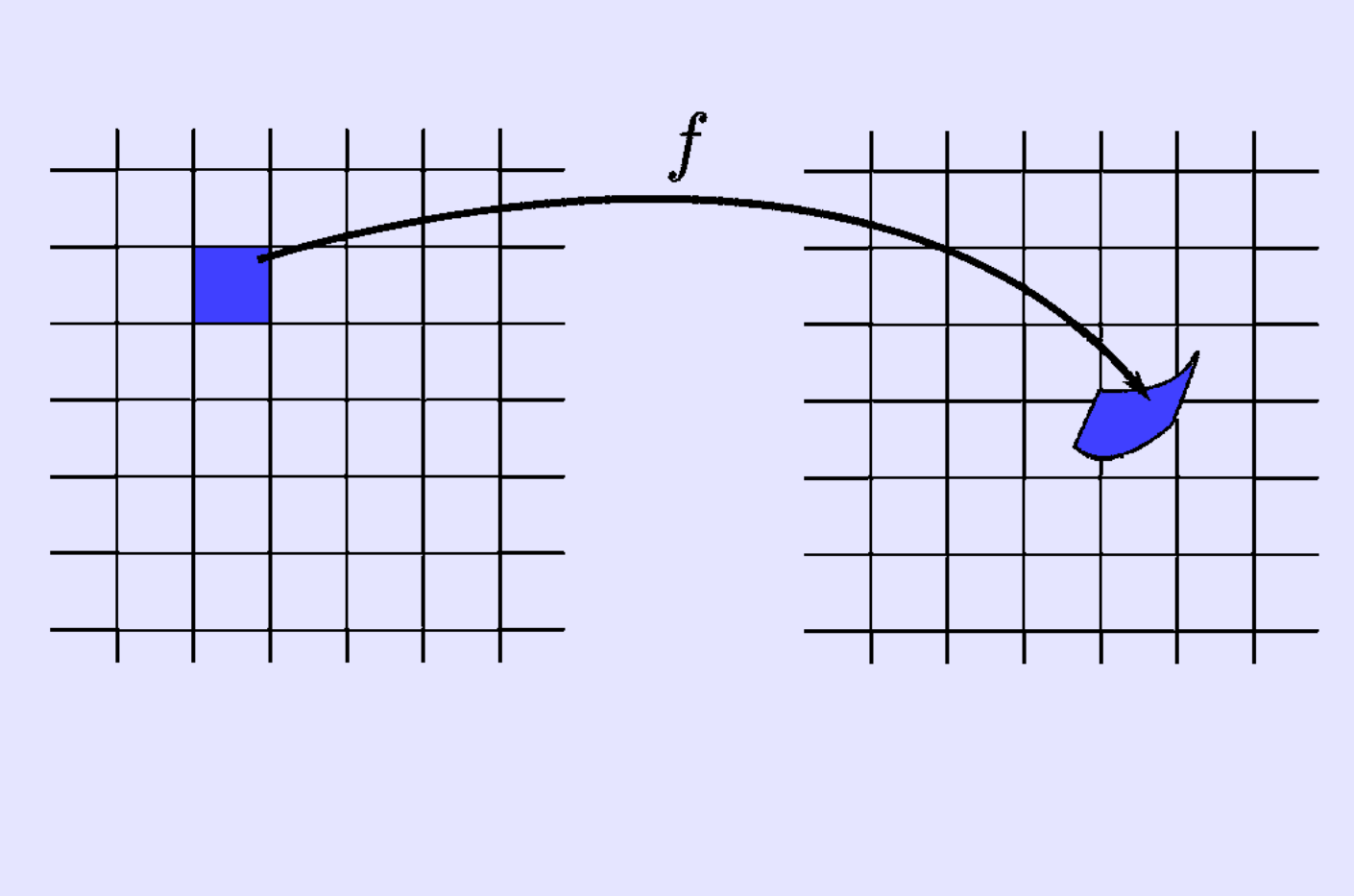

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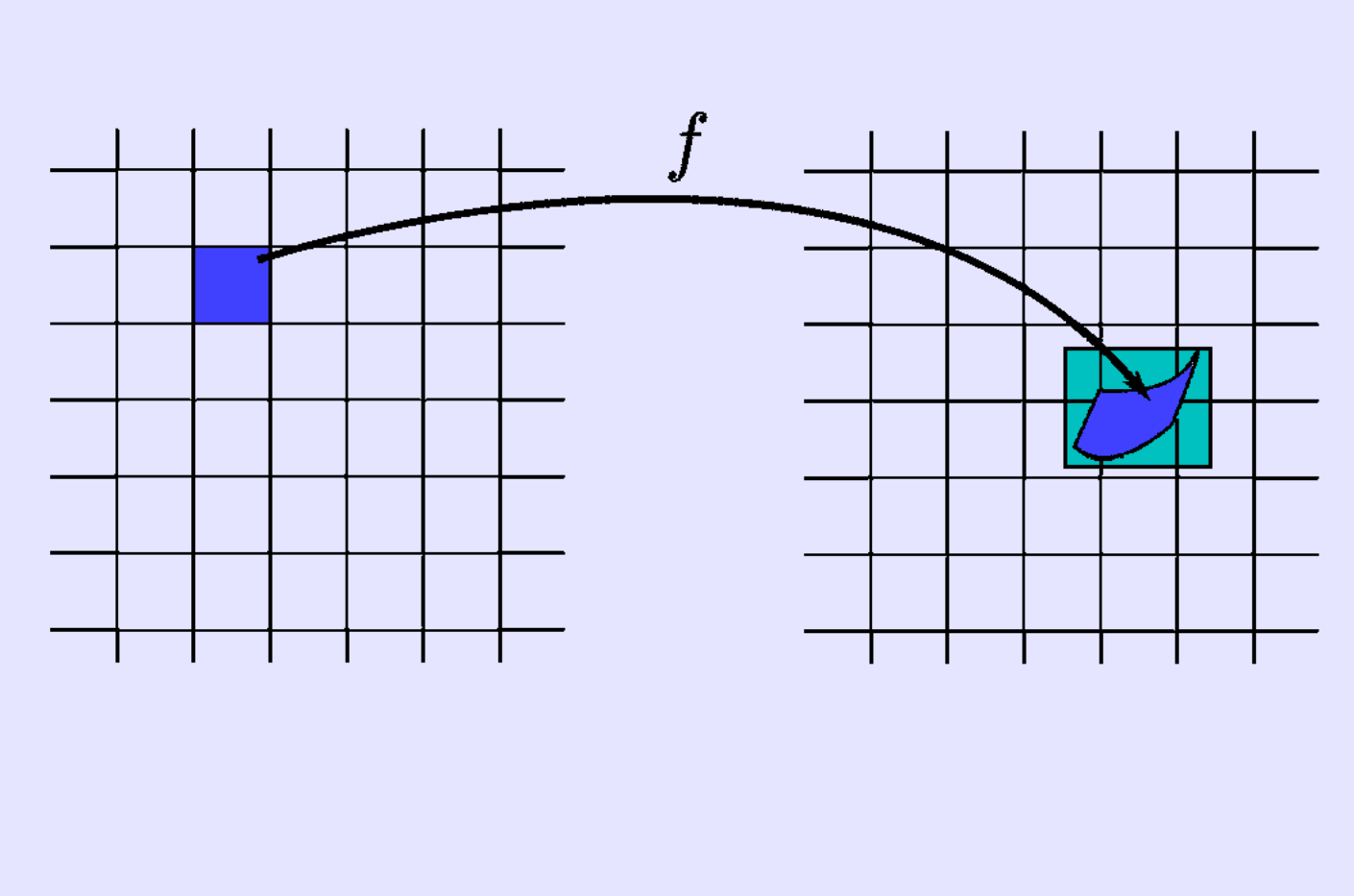

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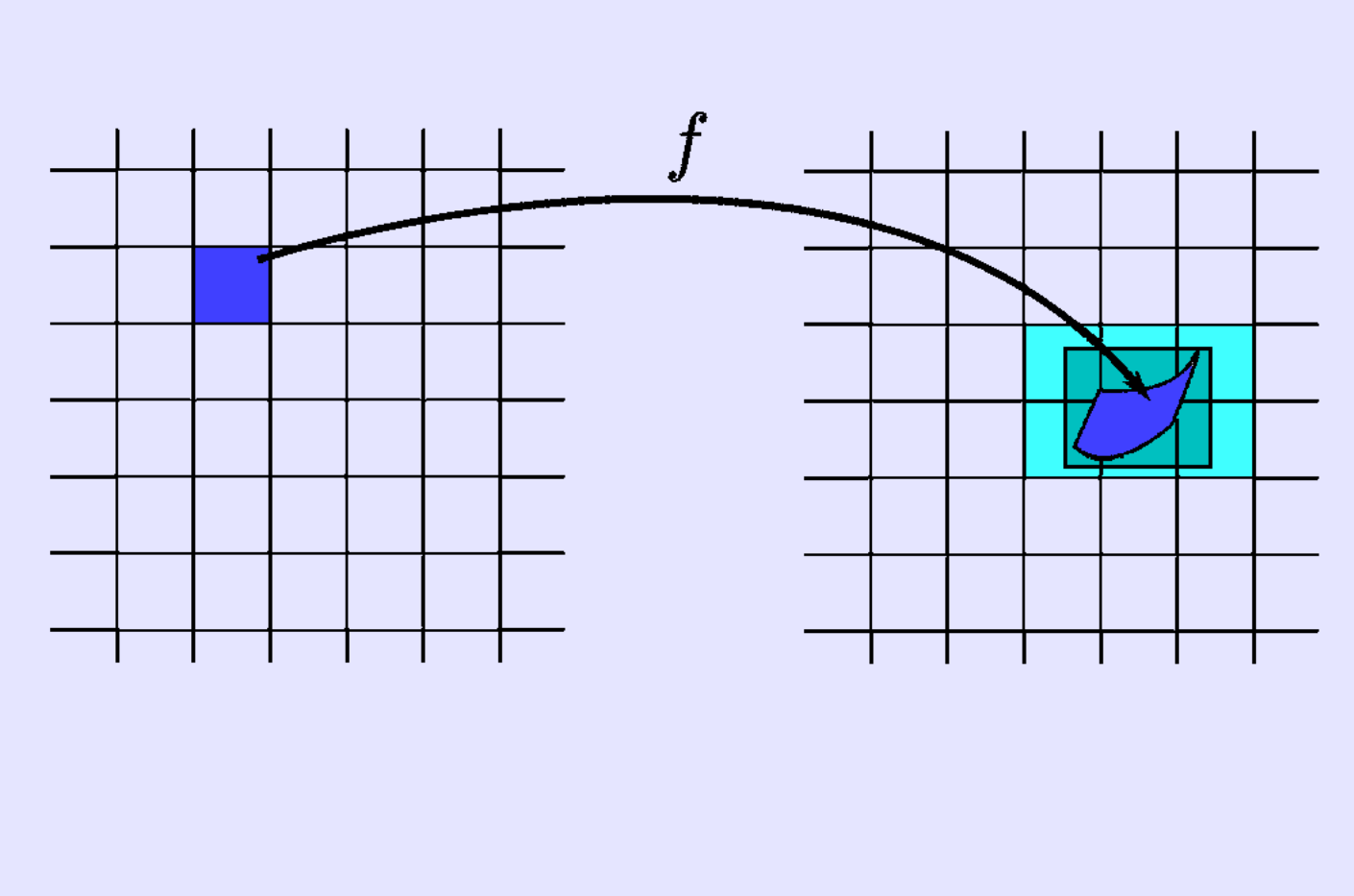

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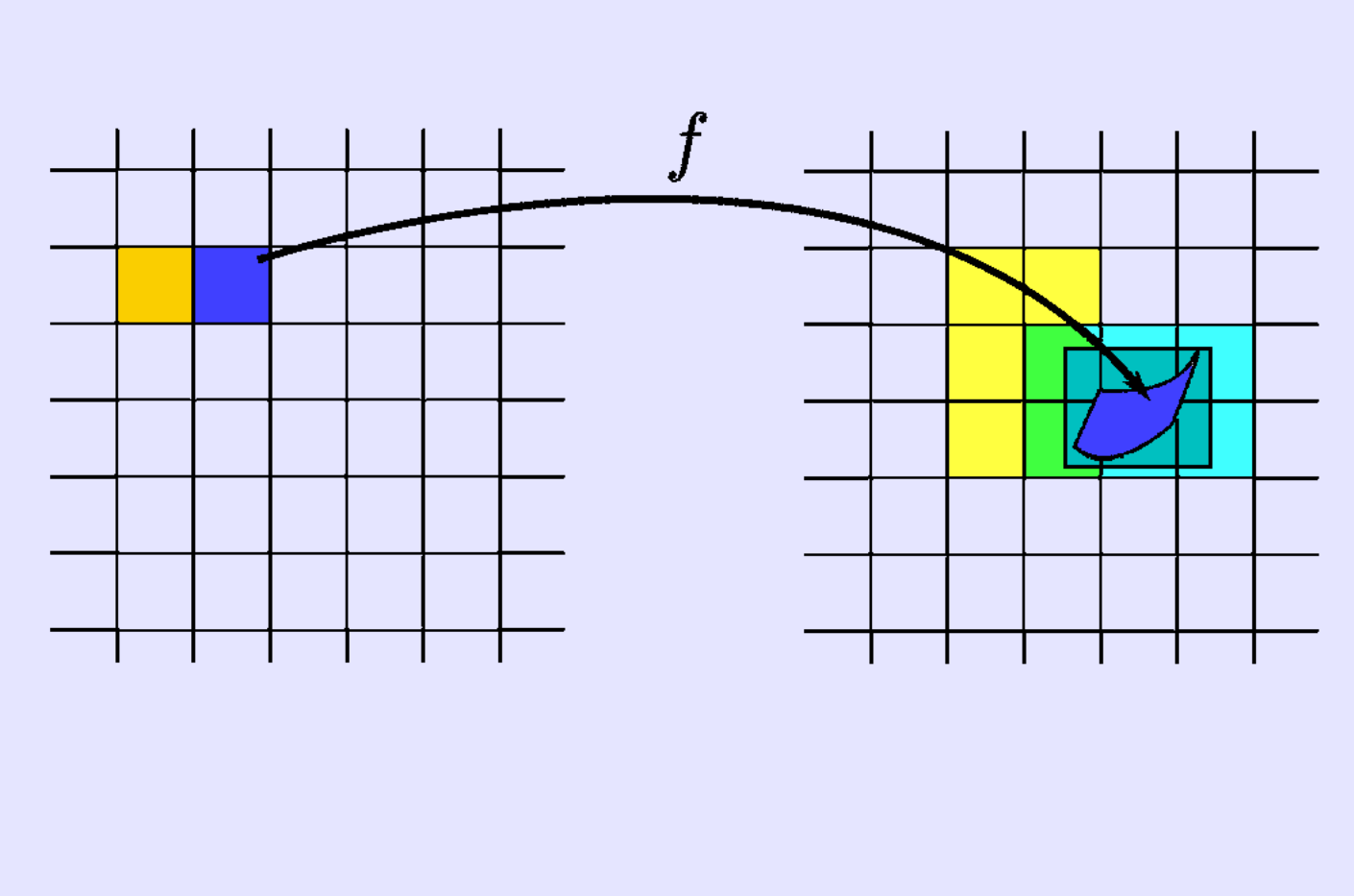

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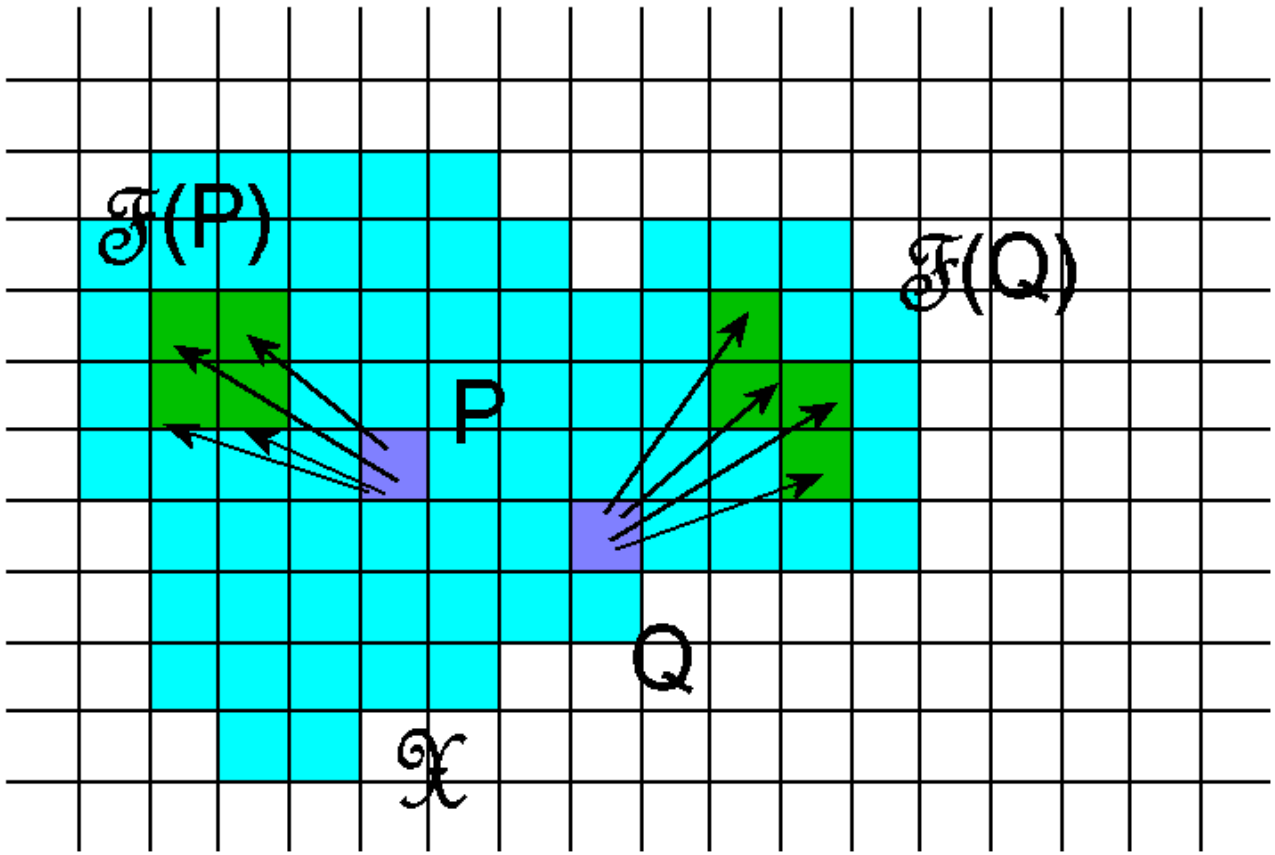

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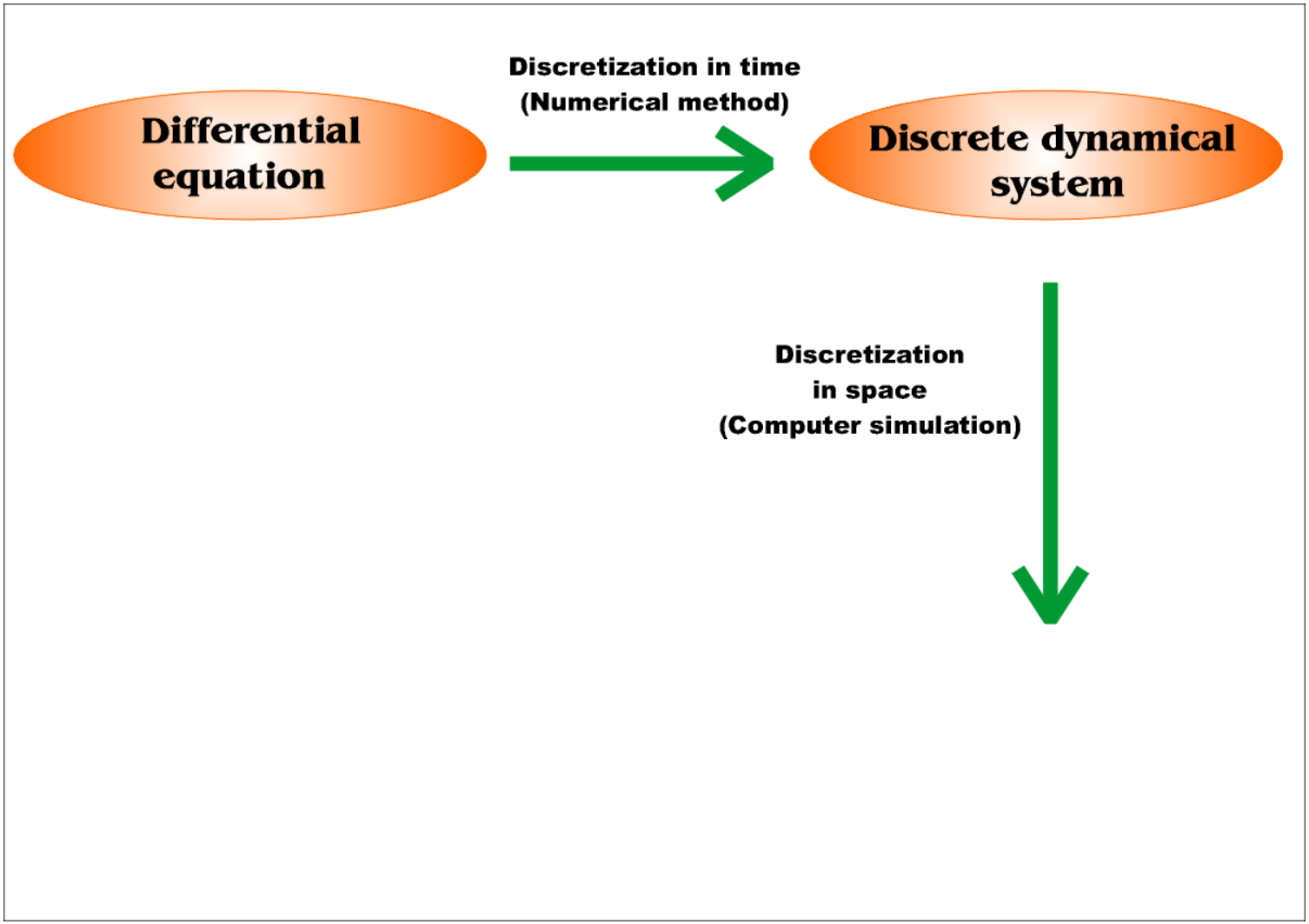
Numerical Analysis of Dynamical Systems



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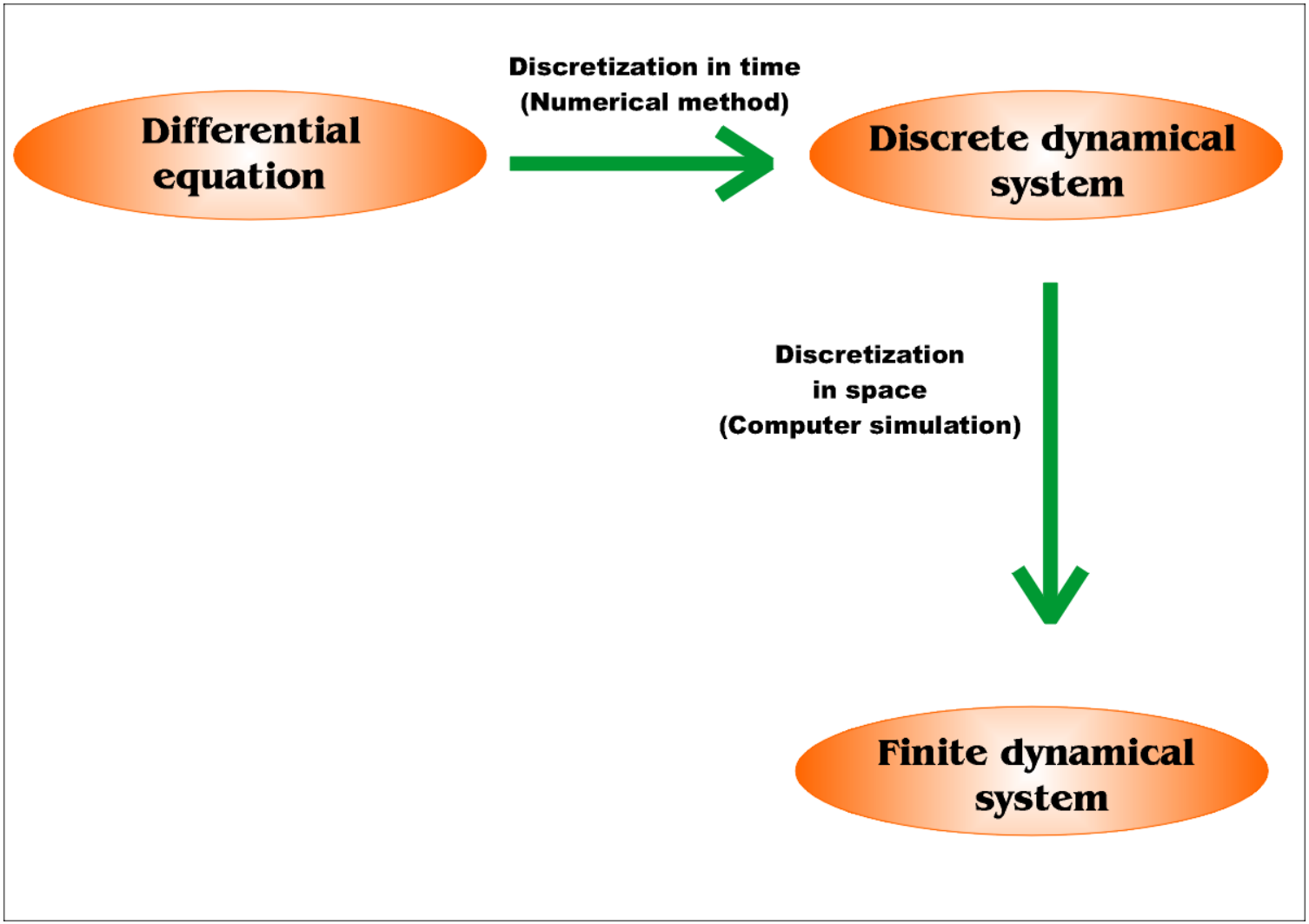

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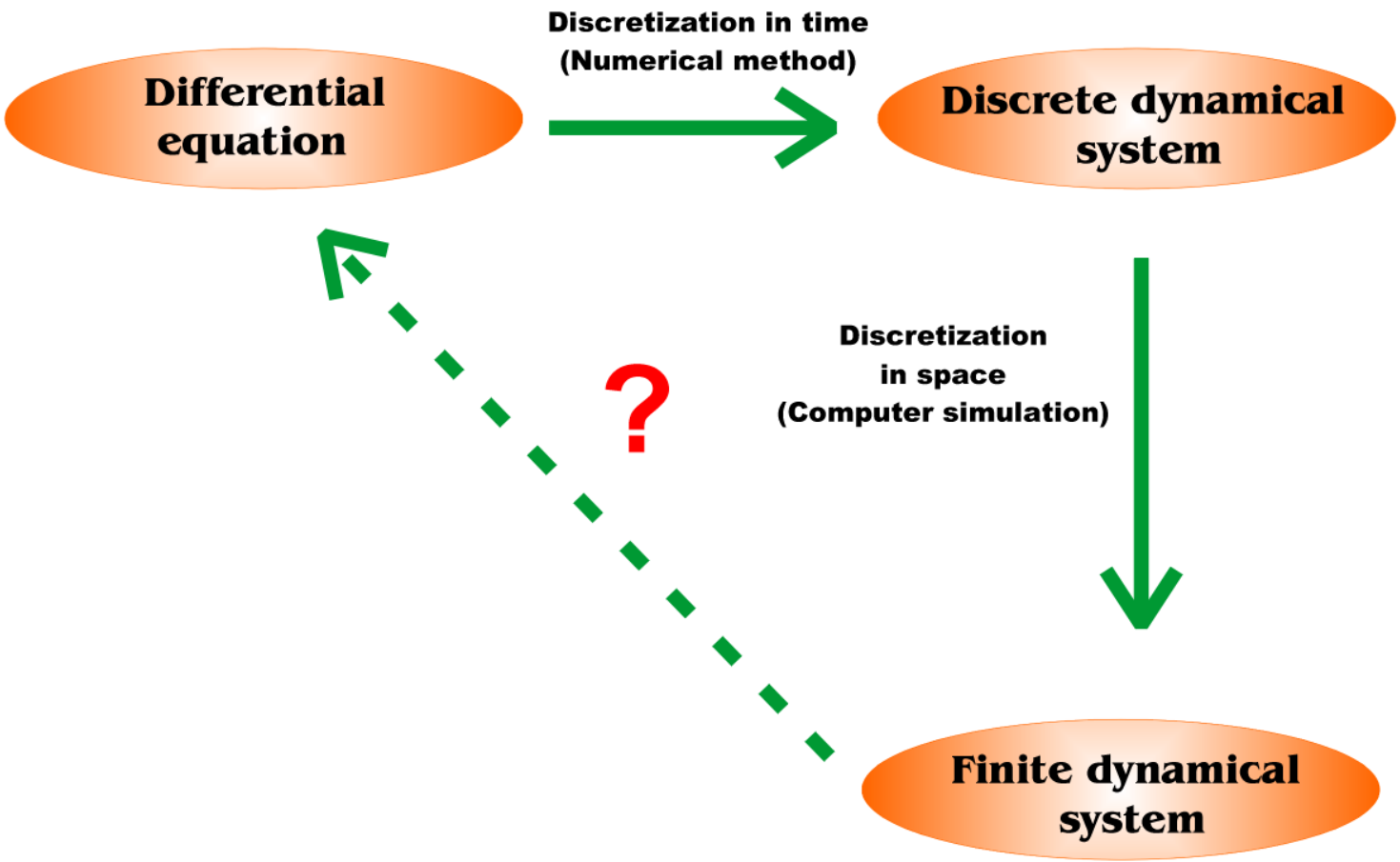

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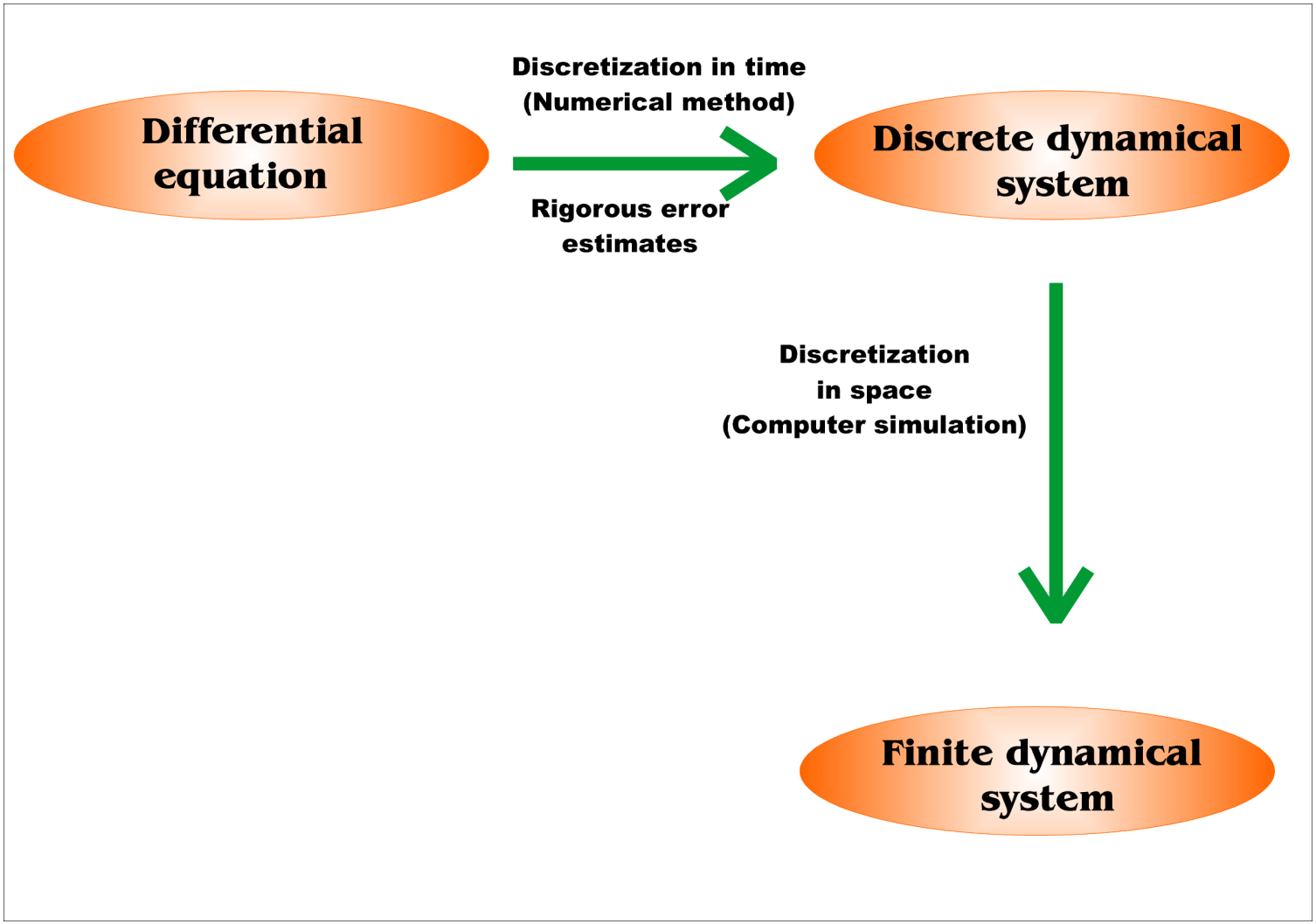

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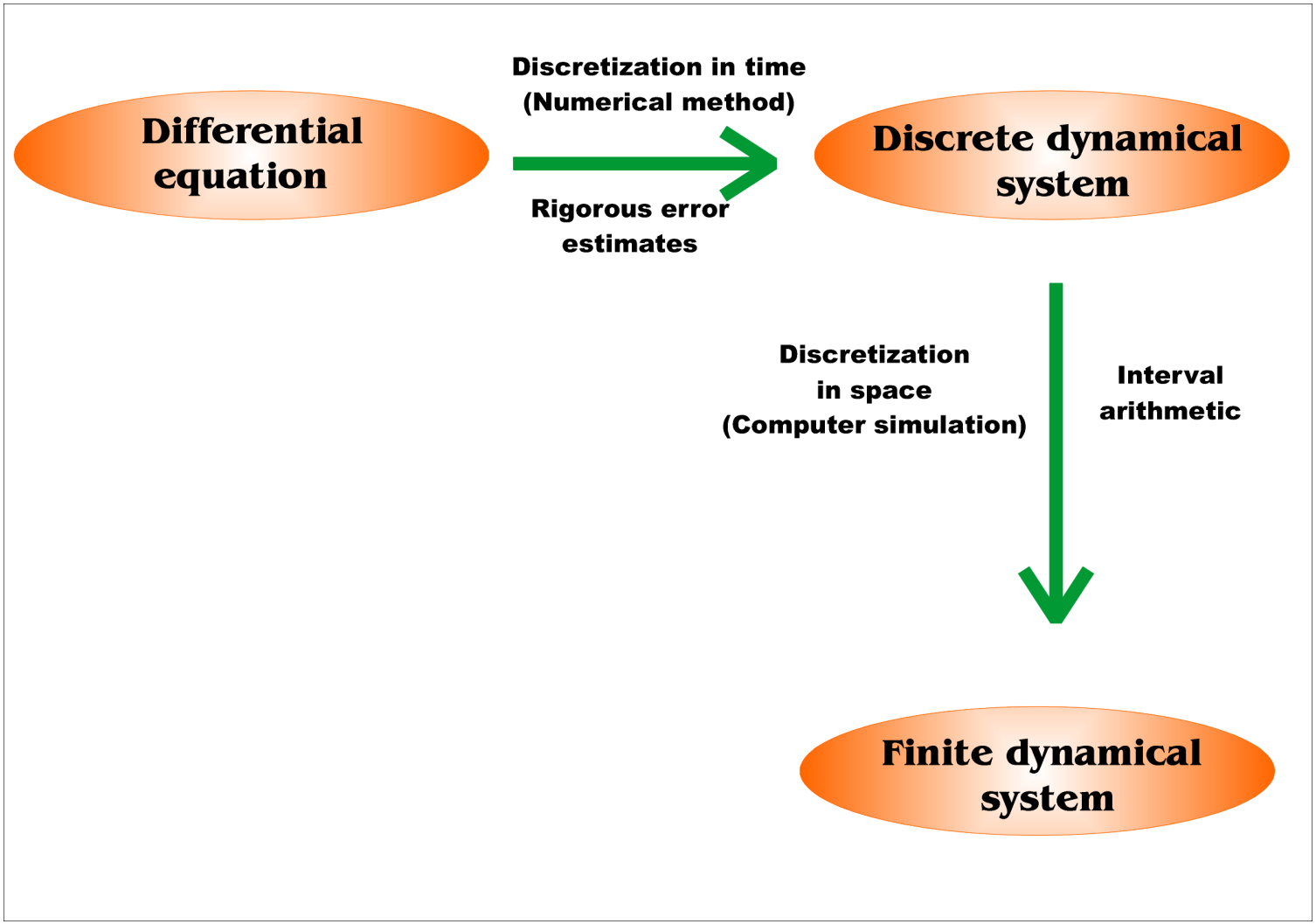

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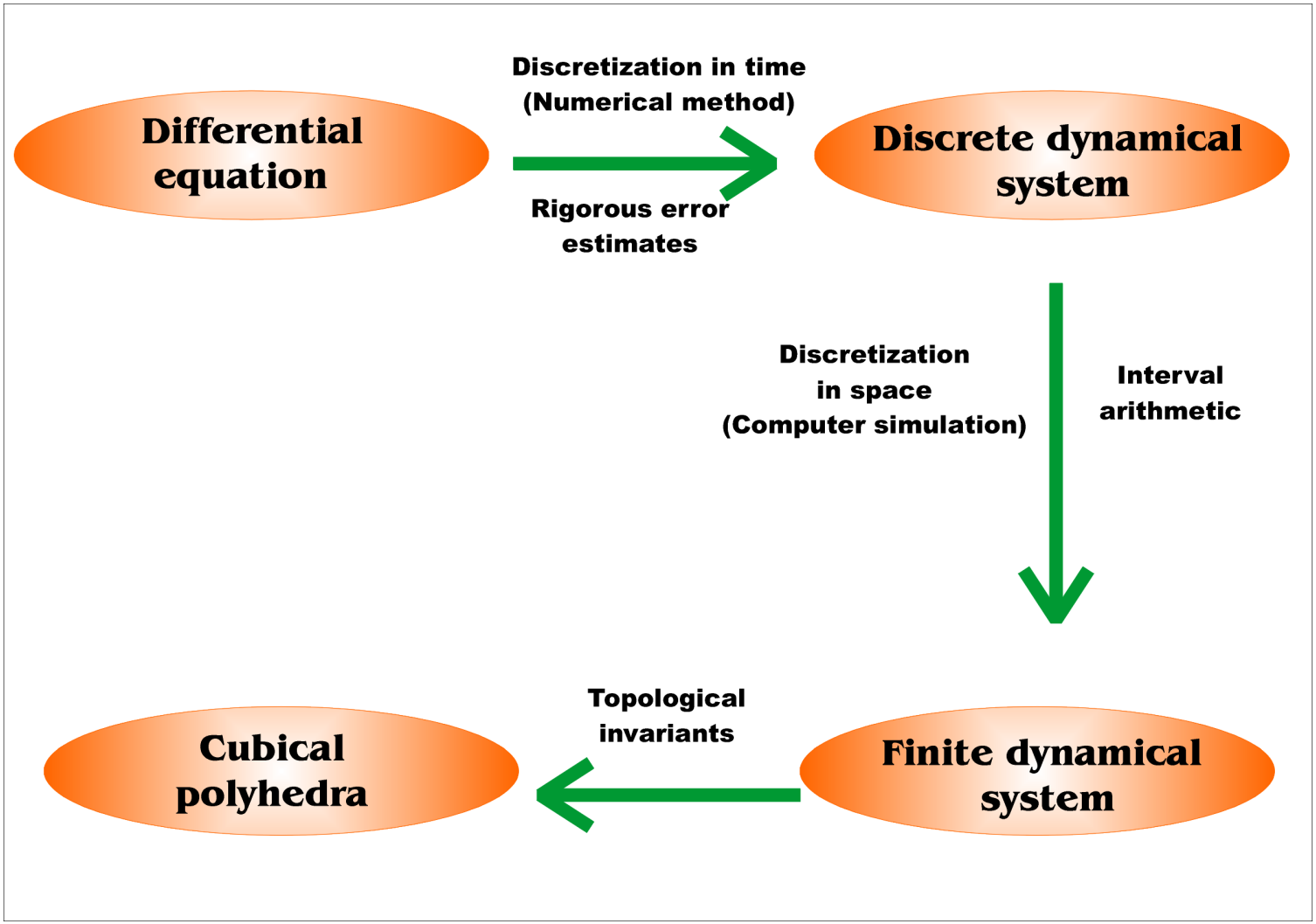

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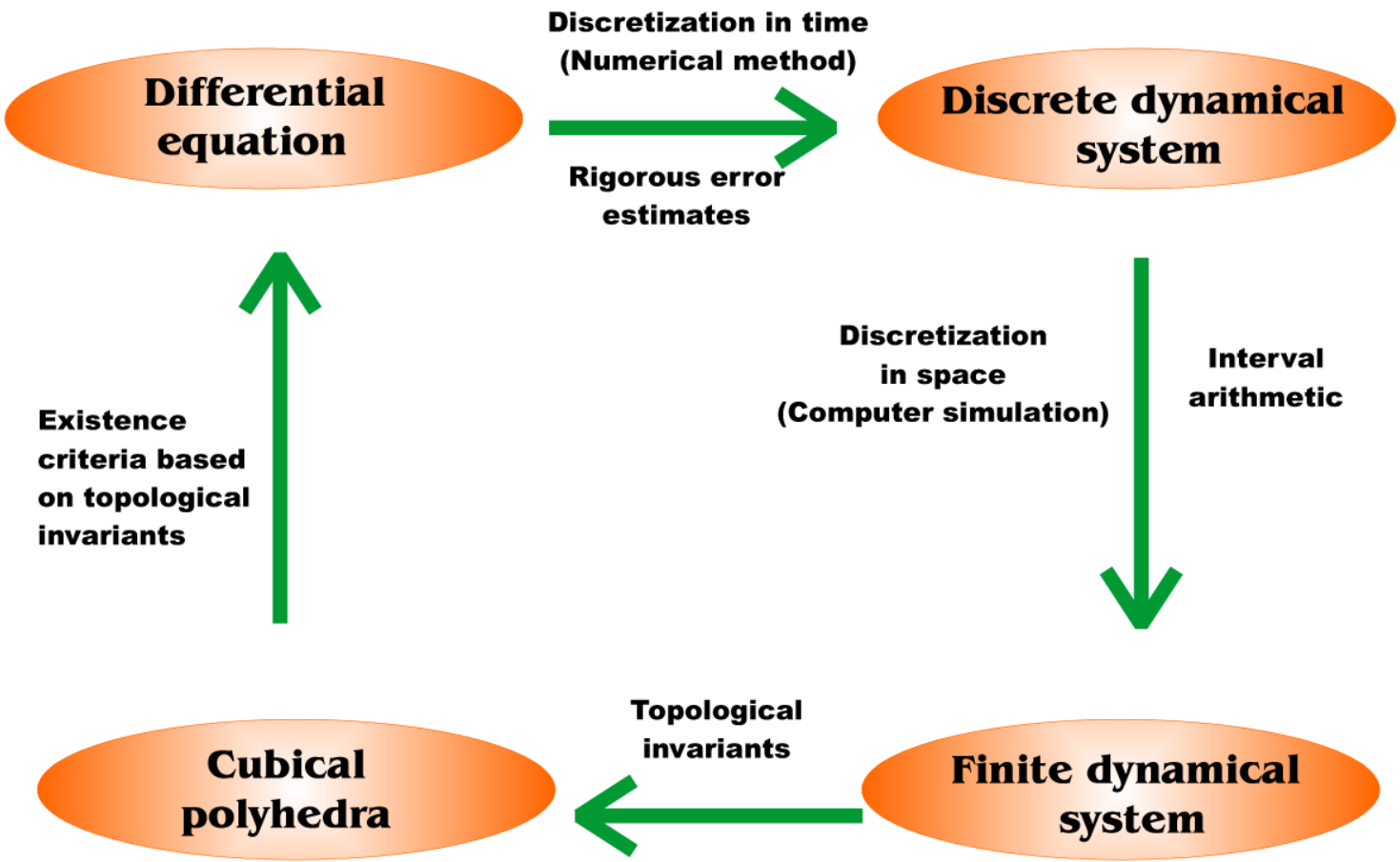

Chaos in Lorenz equations

Theorem. (1995) Consider the Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

and put

$$P := \{(x, y, z) \in \mathbb{R}^3 \mid z = 53\}.$$

For all parameter values in a sufficiently small neighborhood of $(\sigma, R, b) = (45, 54, 10)$, there exists a Poincaré section $N \subset P$ such that the Poincaré map g induced by (1) is Lipschitz and well defined. Furthermore, there exists a $d \in \mathbb{N}$ and a continuous surjection $\rho : \mathrm{Inv}(N, g) \to \Sigma_2$ such that

$$\rho \circ g^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols.

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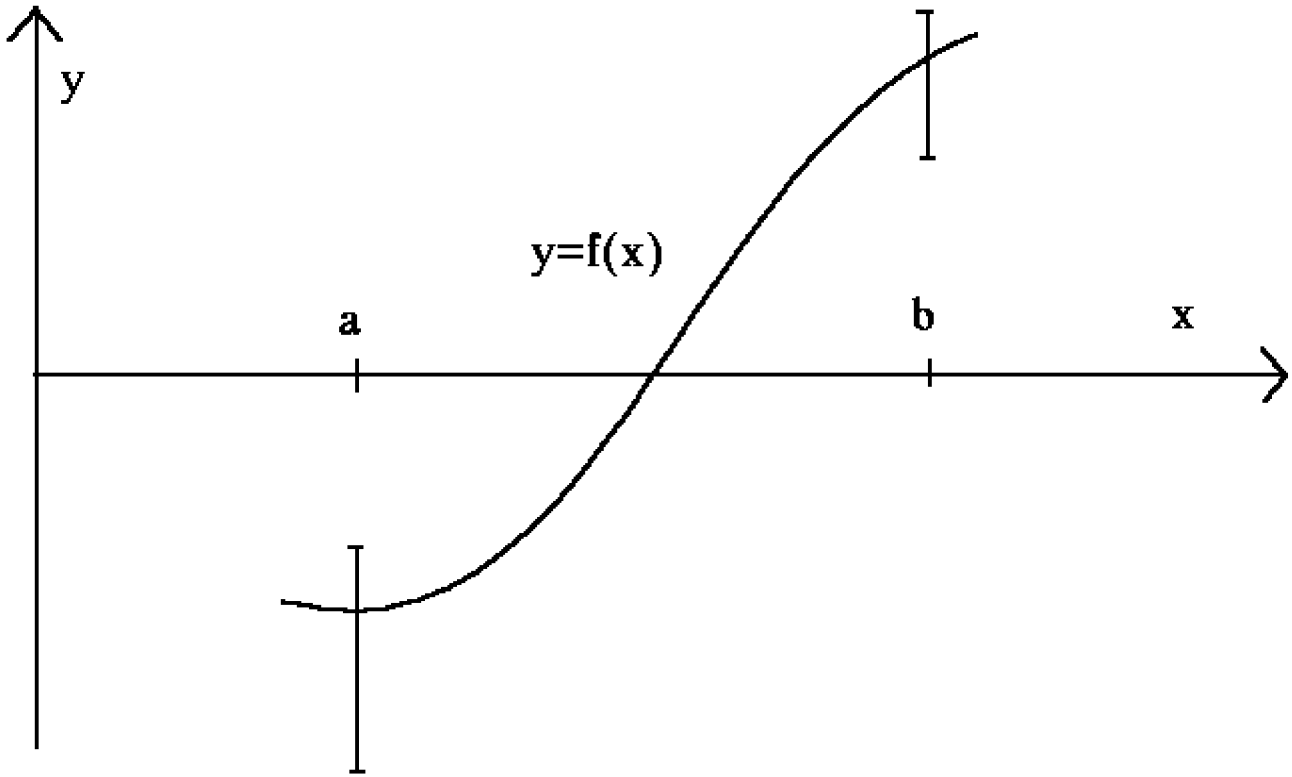
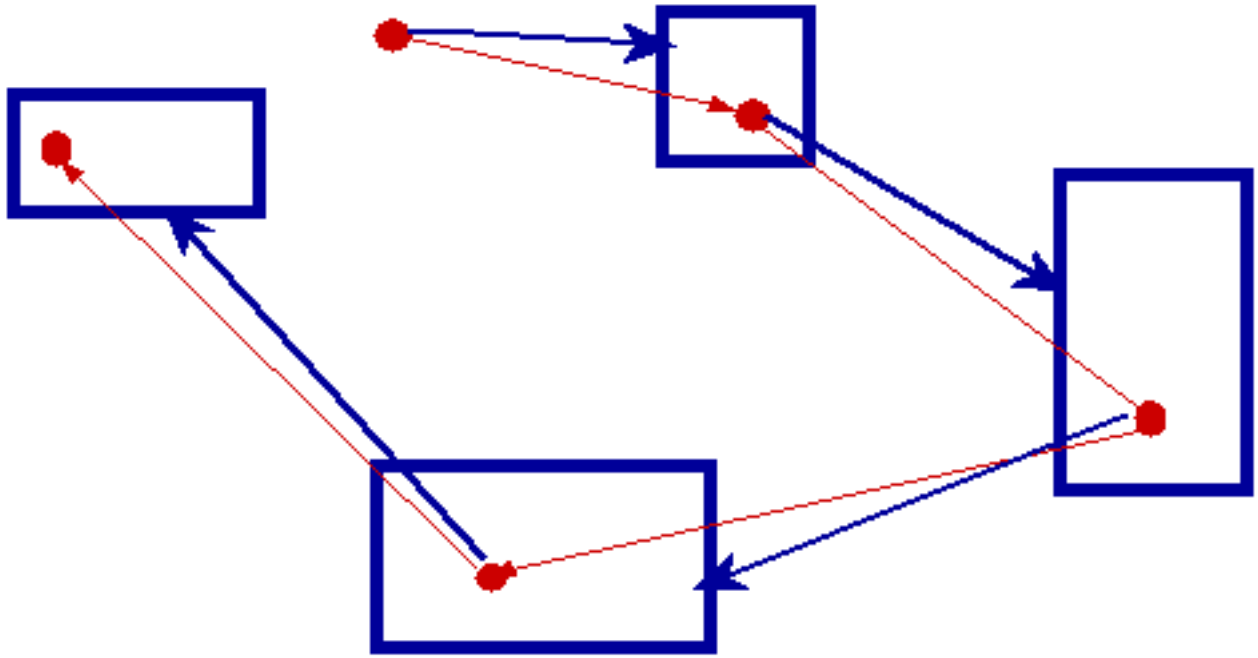


Advanced tool: topology of multivalued maps



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F is upper semicontinuous if $F^{-1}(B)$ is closed for any closed set $B \subset Y$, and it is lower semicontinuous if the set $F^{-1}(U)$ is open for any open set $U \subset Y$.

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Arbitrary functions

- $f : \mathbb{R}^m \multimap \mathbb{R}^n$
- $g : \mathbb{R}^m \multimap \mathbb{R}^n$ — a rational approximation of f such that for $x \in D$ and some $\mathbf{w} \in \mathcal{I}^n$
$$f(x) - g(x) \in \mathbf{w},$$
- then
$$f(\mathbf{x}) \subset [g](\mathbf{x})[+]\mathbf{w}.$$

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- **R.E. Moore**, Interval analysis, *Prentice-Hall* (**1966**).
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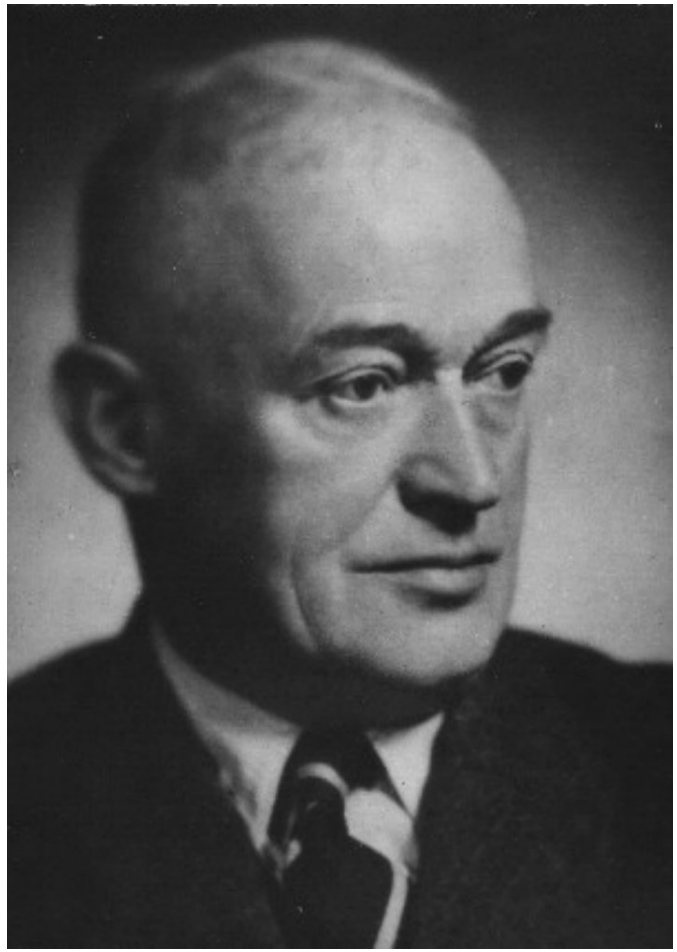
- **K. Mischaikow, MM**, Chaos in the Lorenz equations: a computer assisted proof. Part II: details, *Mathematics of Computation* (**1998**).
- **A.M. Stuart, A.R. Humphries**, Dynamical Systems and Numerical Analysis, *Cambridge Univ. Press* (**1998**).

Outline

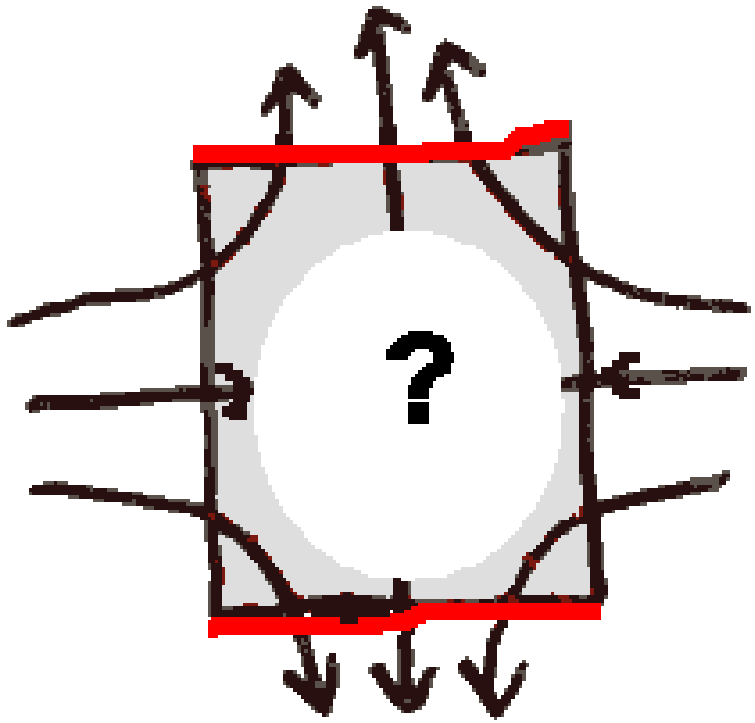
- Dynamical systems
- Rigorous numerics of dynamical systems
- **Homological invariants of dynamical systems**
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Ważewski Theorem



Tadeusz Ważewski, 1896-1972

Ważewski Theorem



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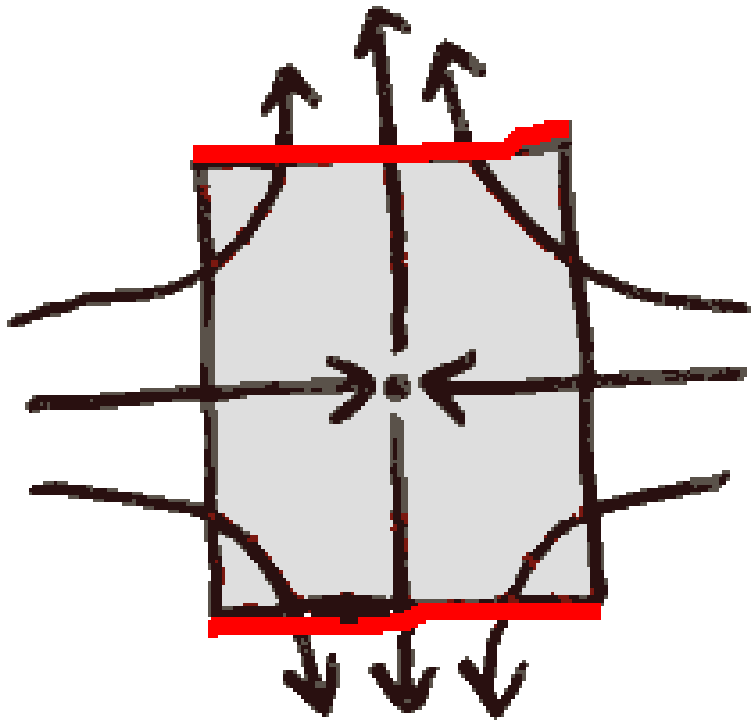

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A compact set $S \subset X$ is called an isolated invariant set if there exists an isolating neighborhood N such that $S = \operatorname{Inv}(N, \varphi)$.

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Theorem. (Conley and students, 1978)

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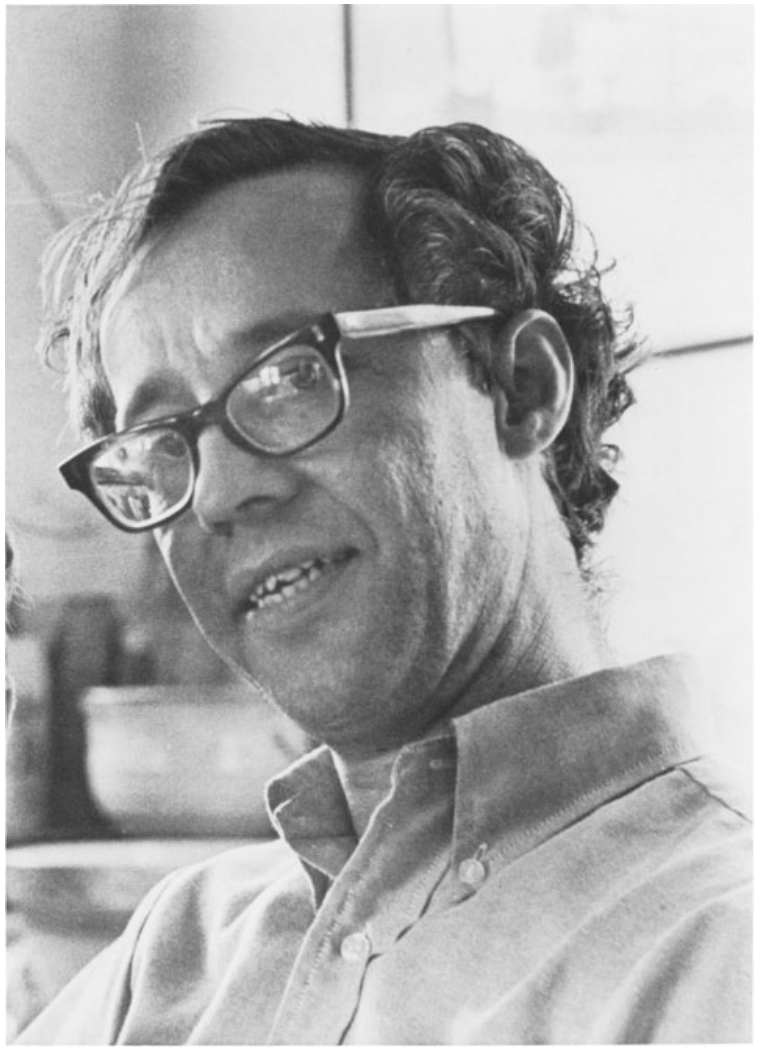
The cohomological Conley index of S and N is

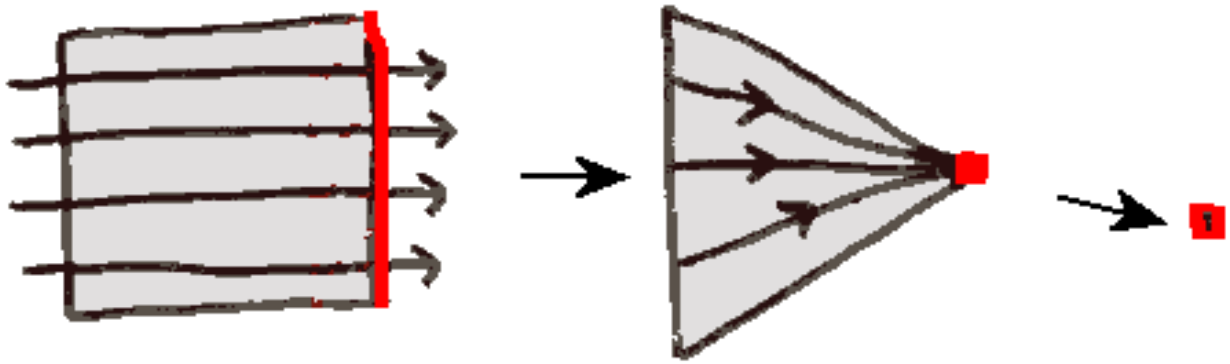
$$\mathrm{Con}^*(N, \varphi) := \mathrm{Con}^*(S, \varphi) := H^*(M, M^-).$$

Charles Conley



Charles Conley
1933-1984

An example



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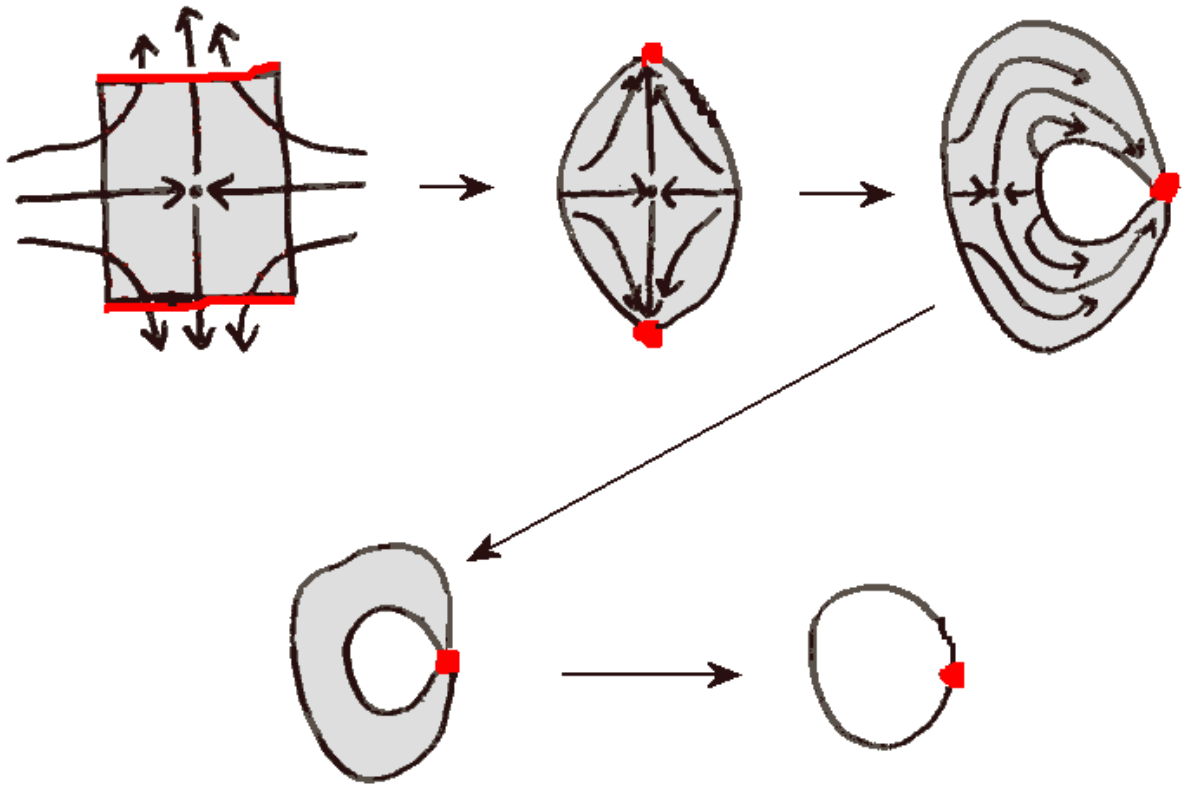

Main properties

Theorem. (Conley and students, 1978)

- **Ważewski property:**
$$\mathrm{Con}^*(N,\varphi) \neq 0 \;\Rightarrow\; \mathrm{Inv}(N,\varphi) \neq \emptyset.$$
- **Hopf property:** If $\chi(\mathrm{Con}^*(N,\varphi)) \neq 0$ then there exists an $x \in N$ such that $\varphi(x) = \{x\}$.
- **Additivity:** If $S = S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$ then
$$\mathrm{Con}^*(S,\varphi) = \mathrm{Con}^*(S_1,\varphi) \oplus \mathrm{Con}^*(S_2,\varphi).$$
- **Homotopy invariance:** If N is an isolating neighborhood for a family of flows φ_t continuously depending on t then
$$\mathrm{Con}^*(N,\varphi_0) = \mathrm{Con}^*(N,\varphi_1).$$

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- Then, N is called an isolating neighborhood (for S).

Index pairs

A pair of compact sets $P = (P_1, P_2)$ is called an index pair for f and an isolated invariant set S iff

(i) (positive relative invariance)

$$f(P_2) \cap P_1 \subset P_2$$

(ii) (exit set)

$$P_1 \cap \mathrm{cl}(f(P_1) \setminus P_1) \subset P_2$$

(iii) (isolation)

$$S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2)$$

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A pair of compact sets $P = (P_1, P_2)$ is called an index pair for f and an isolated invariant set S iff

(i) (positive relative invariance)

$$f(P_2) \cap P_1 \subset P_2$$

(ii) (exit set)

$$P_1 \cap \mathrm{cl}(f(P_1) \setminus P_1) \subset P_2$$

(iii) (isolation)

$$S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2)$$

Index pairs

Proposition. If pair $P = (P_1, P_2)$ of compact subsets of an isolating neighborhood N satisfies

$$f(P_2) \cap P_1 \subset P_2$$
$$P_1 \setminus f^{-1}(N) \subset P_2$$
$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \setminus P_2).$$

then P is an index pair for f and $\operatorname{Inv}(N, f)$.

$H^*(P_1, P_2)$ is not an invariant.

Leray Functor 78

- $\mathcal{E}$ - a category

Leray Functor

- $\mathcal{E}$ - a category
- the category of endomorphisms of $\mathcal{E}$:
  - Objects: pairs (E, e), where $A \in \mathcal{E}$ and $e \in \mathcal{E}(E, E)$
  - Morphisms: $\psi\,(E_1, , e_1) \to (E_2, e_2)$ iff $\psi \in \mathcal{E}(E_1, E_2)$ and $\psi e_1 = e_2 \psi$

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Index quadruples and index maps

A quadruple $P = (P_1, P_2, \bar{P}_1, \bar{P}_2)$ is an index quadruple for f and S if (P_1, P_2) is an index pair for f and S and $(\bar{P}_1, \bar{P}_2)$ is a topological pair such that the map

$$f_P : (P_1, P_2) \ni x \to f(x) \in (\bar{P}_1, \bar{P}_2)$$
$$\iota_P : (P_1, P_2) \ni x \to x \in (\bar{P}_1, \bar{P}_2)$$

are well defined and ι_P is an excision (induces an isomorphism in cohomology)

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Given an index quadruple, we define the index map as the composition

$$I_P := H^*(f_{P\bar{P}}) \circ H^*(\iota_P)^{-1}$$

The Conley index for discrete dynamical systems

Theorem. (MM,1990,2005) For every isolating neighborhood N of f there exists an index quadruple P such that

$$\mathrm{Inv}(N, f) \subset P_1 \subset \bar{P}_1 \subset N.$$

Moreover, if P and Q are two such quadruples, then

$$L(H^*(P_1, P_2), I_P) \cong L(H^*(Q_1, Q_2), I_Q).$$

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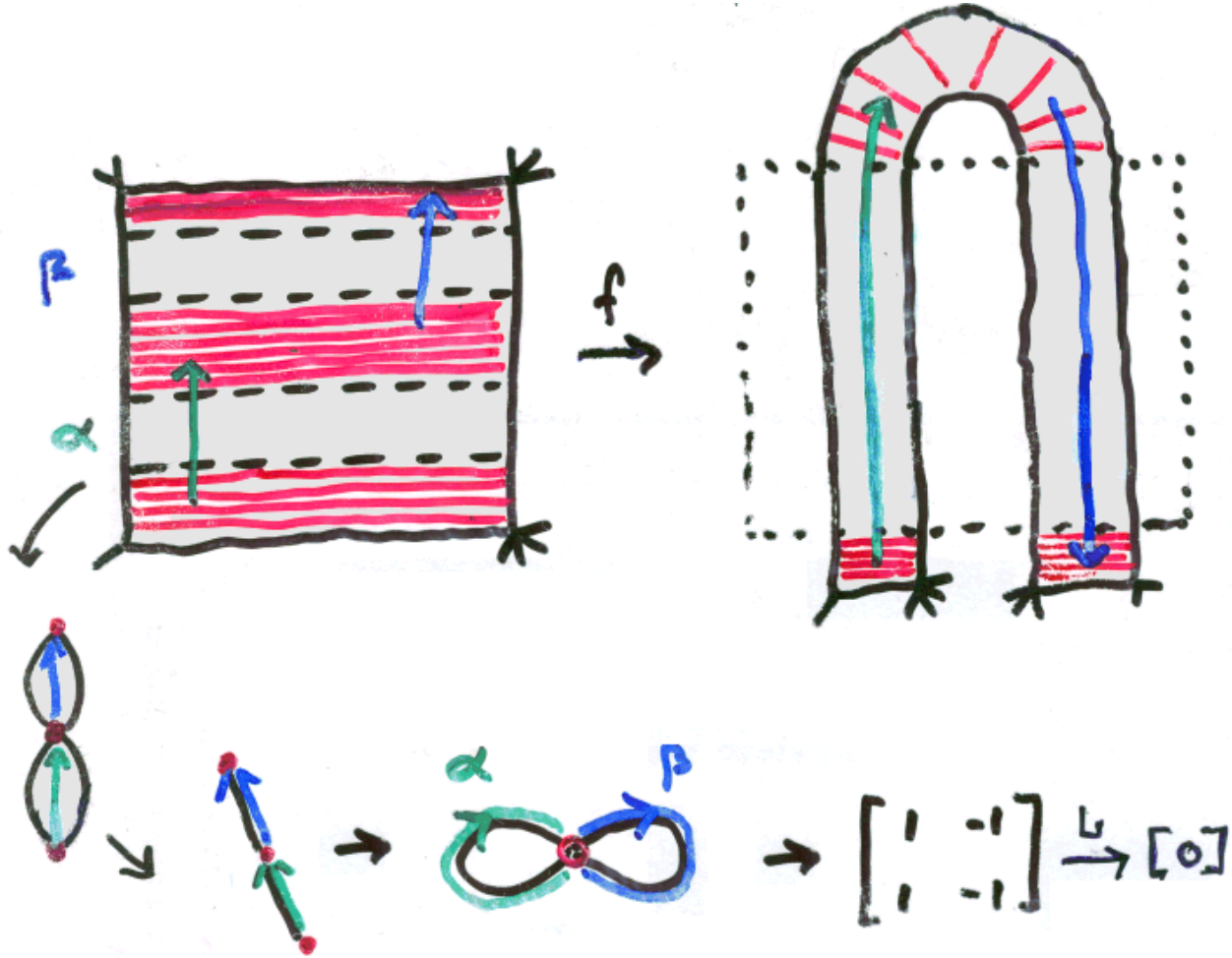
The Conley index of f in N is

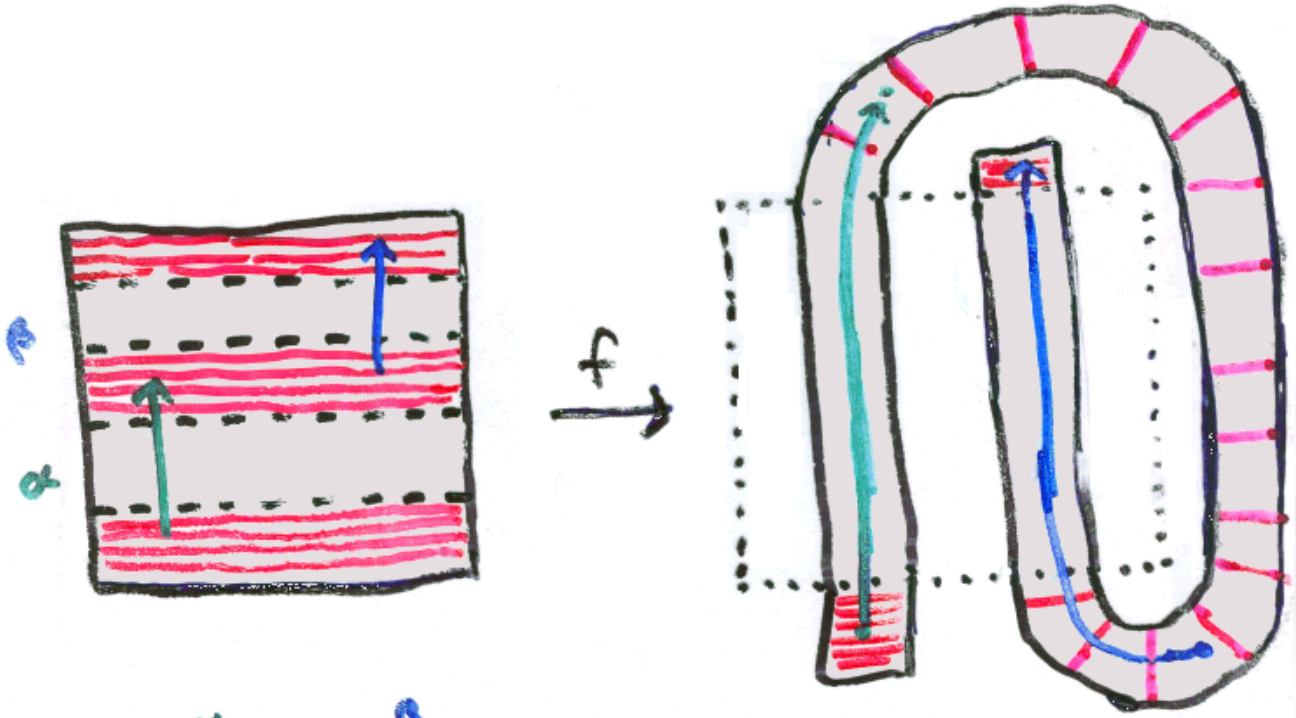
$$(CH^*(N, f), \chi(N, f)) := L(H^*(P_1, P_2), I_P).$$

- J.W. Robbin, D. Salamon, 1988 - shape theory, inverse limit functor
- MM, 1990 - cohomology, Leray functor
- A. Szymczak, 1995 - homotopy, Szymczak functor (most general)
- J. Franks, D. Richeson, 2000 - a reformulation of Szymczak construction in terms of shift equivalence

An example



An example



$$\to \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{L} [2]$$

Main properties

Theorem.

- **Ważewski property** (J.W. Robbin, D. Salamon, 1988):
$$\mathrm{Con}^*(N, f) \neq 0 \;\Rightarrow\; \mathrm{Inv}(N, f) \neq \emptyset.$$
- **Lefschetz property** (MM, 1989): If $\Lambda(\chi(N, f)) \neq 0$ then there exists an $x \in N$ such that $f(x) = x$.
- **Additivity:**(J.W. Robbin, D. Salamon, 1988): If $S = S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$ then
$$\mathrm{Con}^*(S, f) = \mathrm{Con}^*(S_1, f) \oplus \mathrm{Con}^*(S_2, f).$$
- **Homotopy invariance:**(J.W. Robbin, D. Salamon, 1988): If N is an isolating neighborhood for a family of flows f_t continuously depending on t then
$$\mathrm{Con}^*(N, f_0) = \mathrm{Con}^*(N, f_1).$$

Discrete vs. continuous case.

Theorem. (MM, 1990) Let $\varphi : X \times \mathbb{R} \to X$ be a flow and for $t \in \mathbb{R}$ let $\varphi_t : X \to X$ be the map defined by

$$\varphi_t(x) := \varphi(x, t).$$

If $S \subset X$ is a compact set, then the following conditions are equivalent.

(i) S is an isolated invariant set with respect to φ,

(ii) S is an isolated invariant set with respect to φ_t for all $t \neq 0$,

(iii) S is an isolated invariant set with respect to φ_t for some $t \neq 0$.

Moreover, if one of the above conditions is satisfied, then for any $t \neq 0$

$$\chi(S, \varphi_t) = \mathrm{id},$$
$$\mathrm{Con}^*(S, \varphi_t) \cong \mathrm{Con}^*(S, \varphi).$$

Conley index and horseshoe dynamics

Given a compact set N and $\alpha \in \{0,1\}^n$ put

$$N_\alpha := \bigcap_{i=0}^{n-1} f^i(N_{\alpha_i})$$

and for $\bar{\alpha} = (\alpha^1, \alpha^2, \ldots \alpha^m)$ with $\alpha^j \in \{0,1\}^n$ put

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Proposition. If N is an isolating neighborhood for f then so is N_α and $N_{\bar{\alpha}}$

Conley index and horseshoe dynamics

Theorem. (K. Mischaikow, MM, 1993) Assume

$$N = N_0 \cup N_1$$

is an isolating neighbourhood for f such that N_0 and N_1 are disjoint compact polyhedra. If for $k = 0, 1$

$$\mathrm{Con}^n(N_k) = \begin{cases} (\mathbf{Q}, \mathrm{Id}) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\chi^*(N_{00,01,11}, f)$, $\chi^*(N_{00,10,11}, f)$ are different from identity then there exists a $d \in \mathbf{N}$ and a continuous surjection

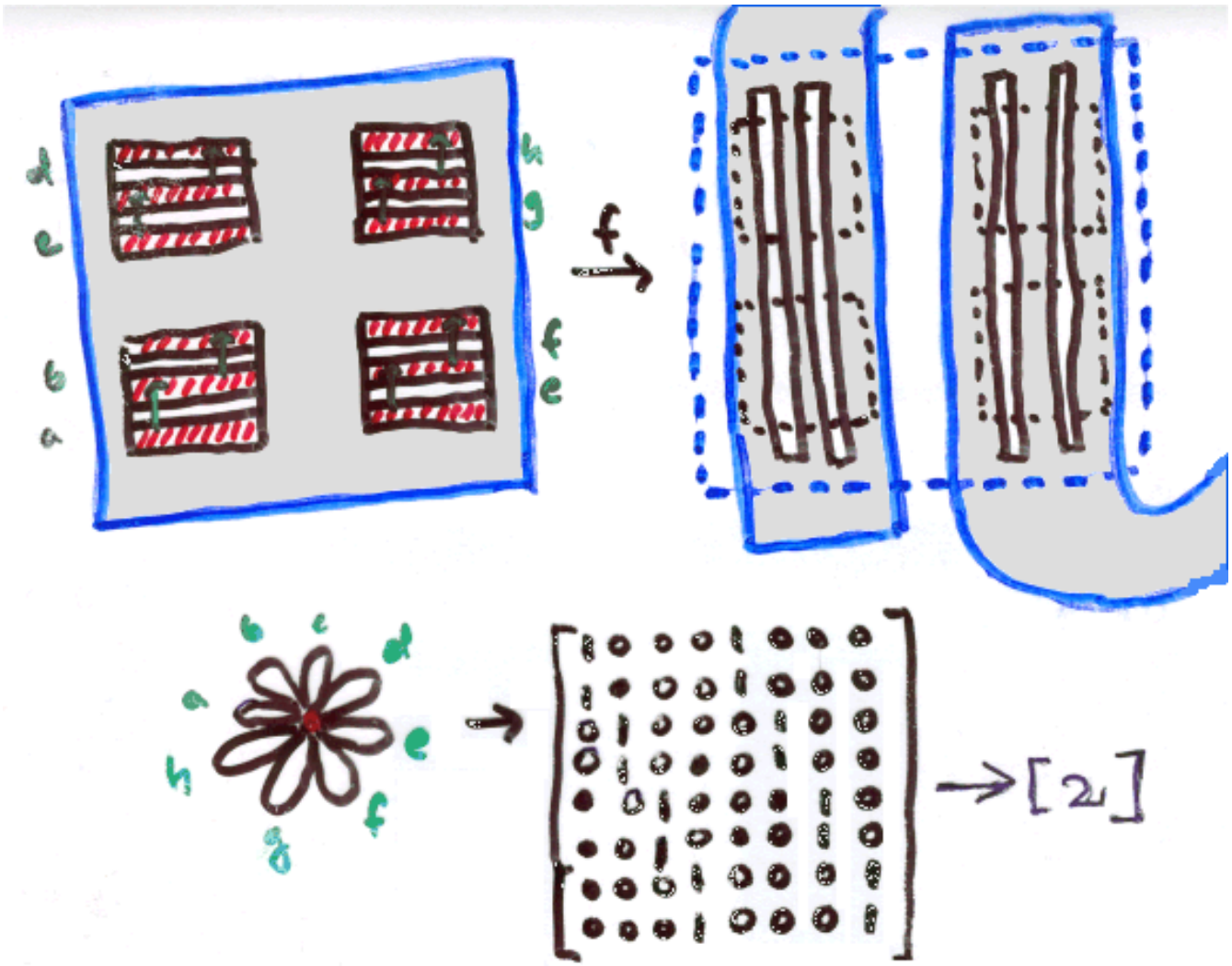
$$\rho : \mathrm{Inv}(N, f) \to \Sigma_2$$

such that

$$\rho \circ f^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols. Moreover, for each periodic sequence $\alpha \in \Sigma_2$ there exists a periodic point $x \in N$ such that $\rho(x) = \alpha$.

An example



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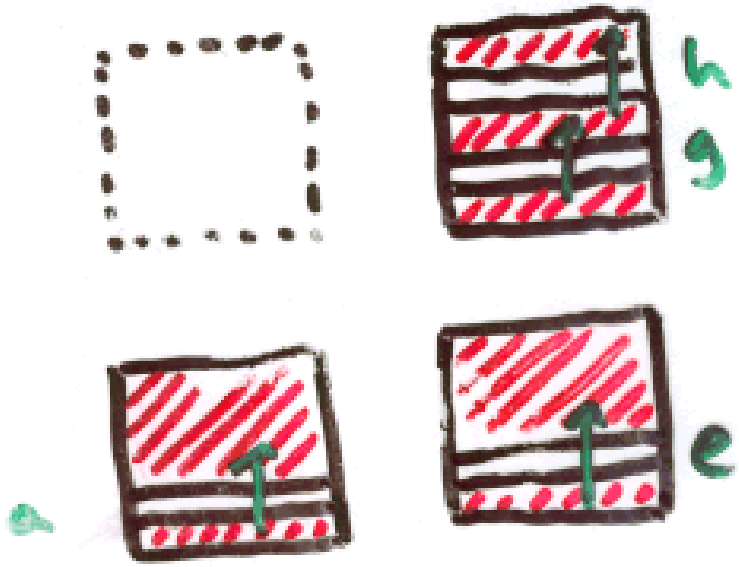
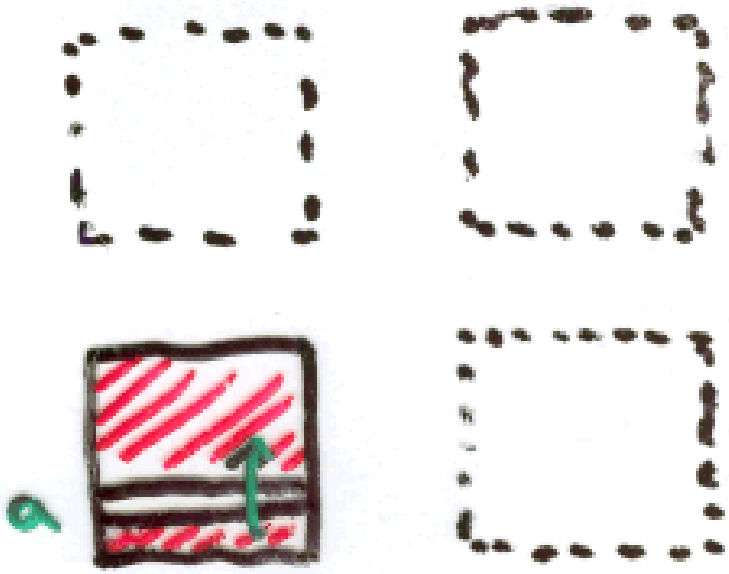


$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

An example



[1]

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Outline

- Dynamical systems
- Rigorous numerics of dynamical systems
- Homological invariants of dynamical systems
- **Computing homological invariants**
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Cubical sets

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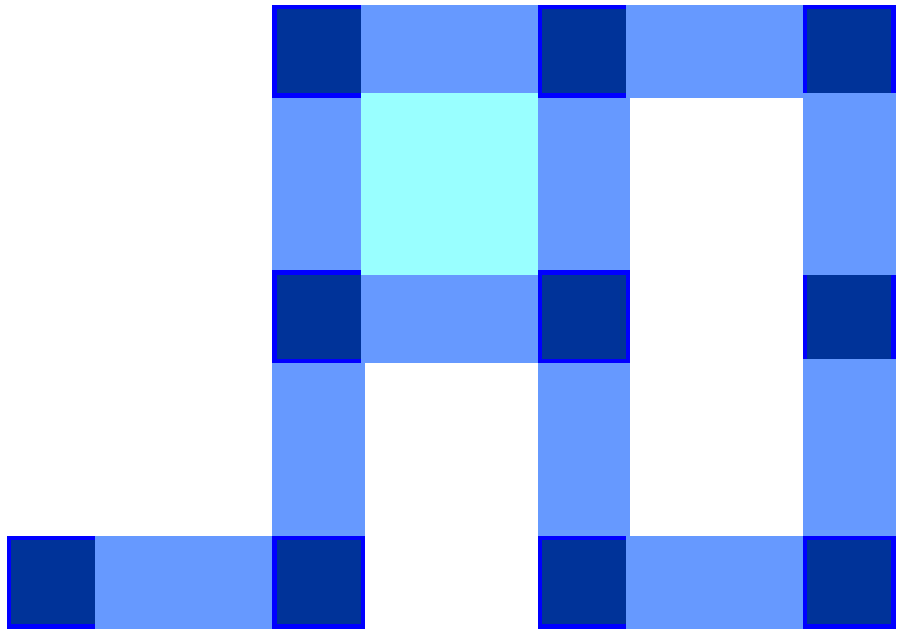
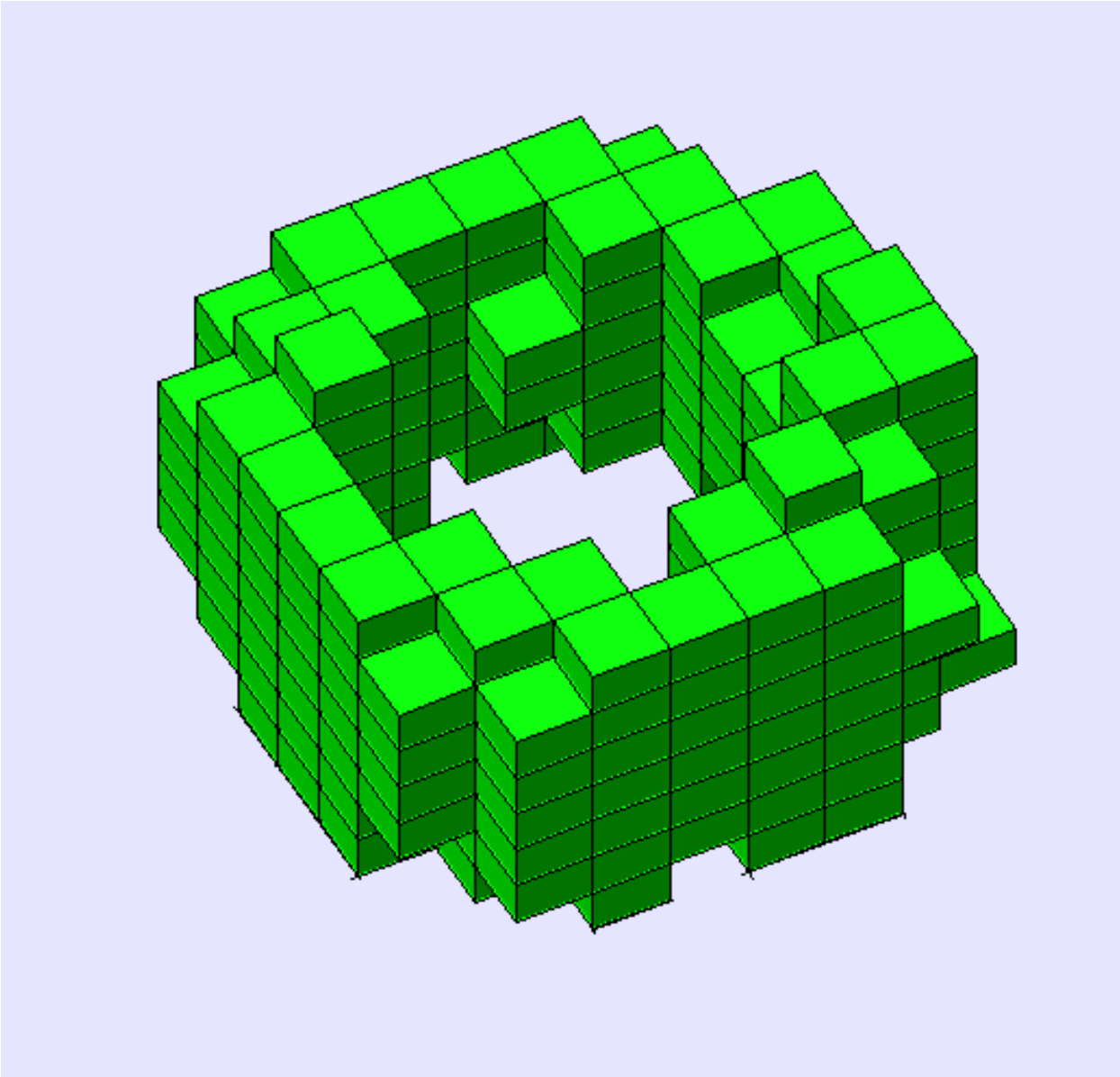
Theorem. (Blass, Holsztyński, 1972) Every polyhedron is homeomorphic to a cubical set.

A cubical set in $\mathbb{R}^2$



A full cubical set in $\mathbb{R}^3$



Combinatorial boundary and interior

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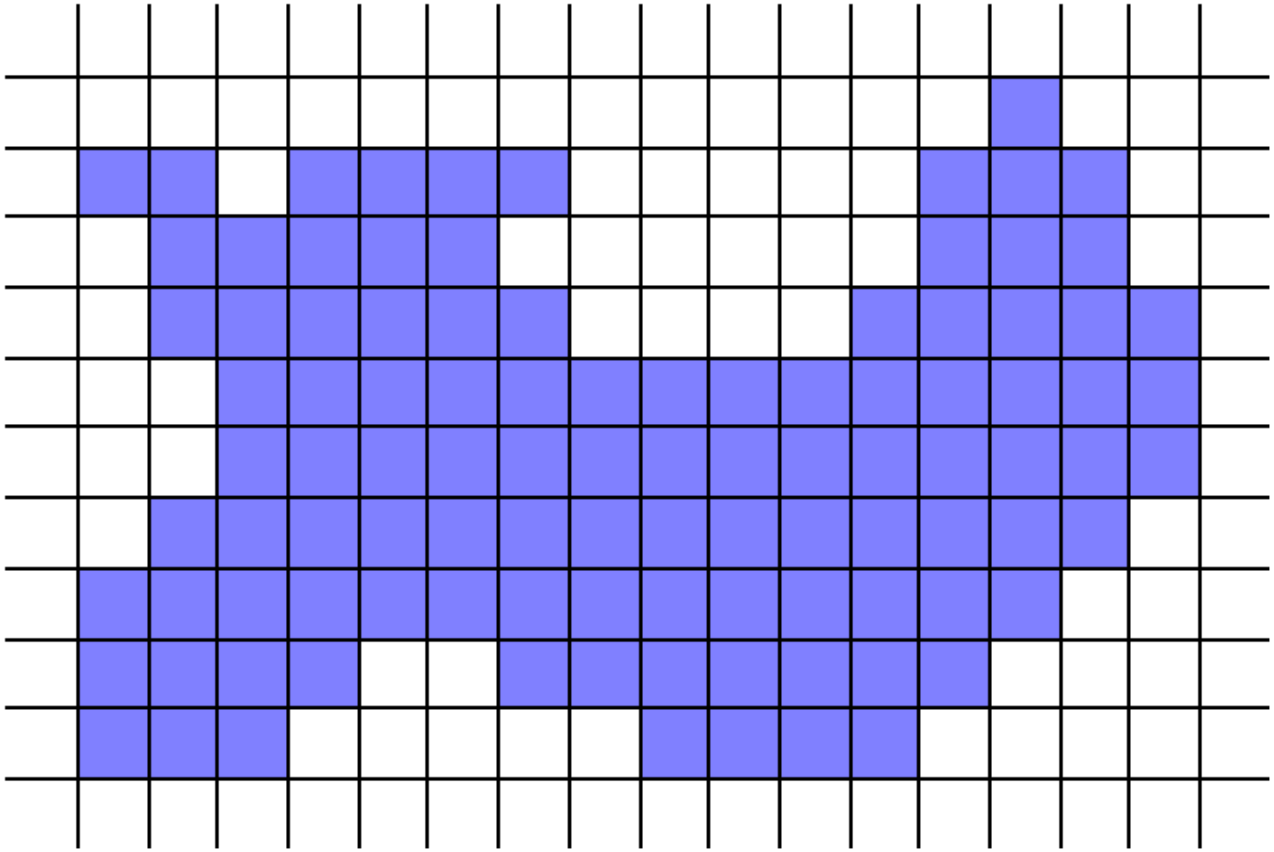
$$\operatorname{int} \mathcal{N} := \{\, Q \in \mathcal{N} \mid o_d(Q) \subset \mathcal{N} \,\},$$
$$\operatorname{bd} \mathcal{N} := \mathcal{N} \setminus \operatorname{int}(\mathcal{N}).$$

Combinatorial boundary and interior



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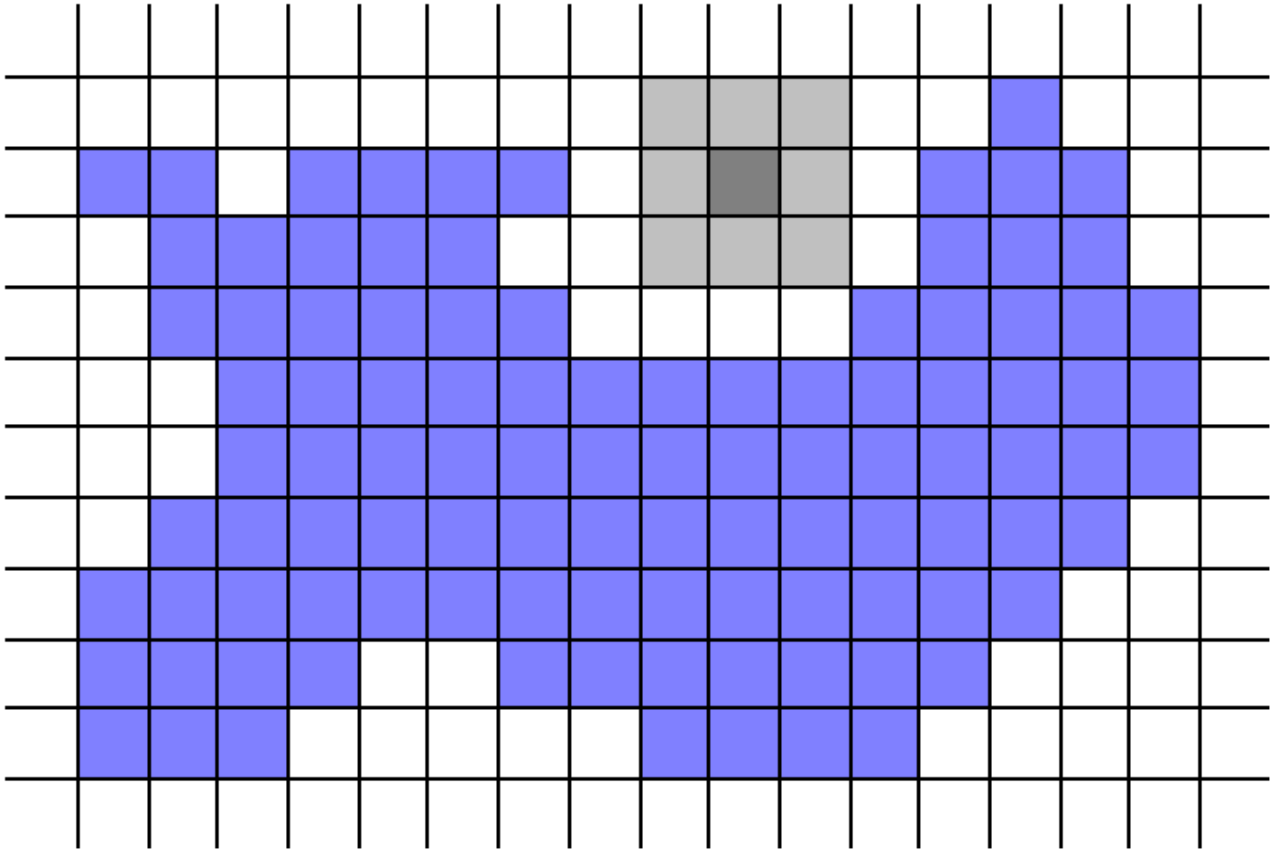

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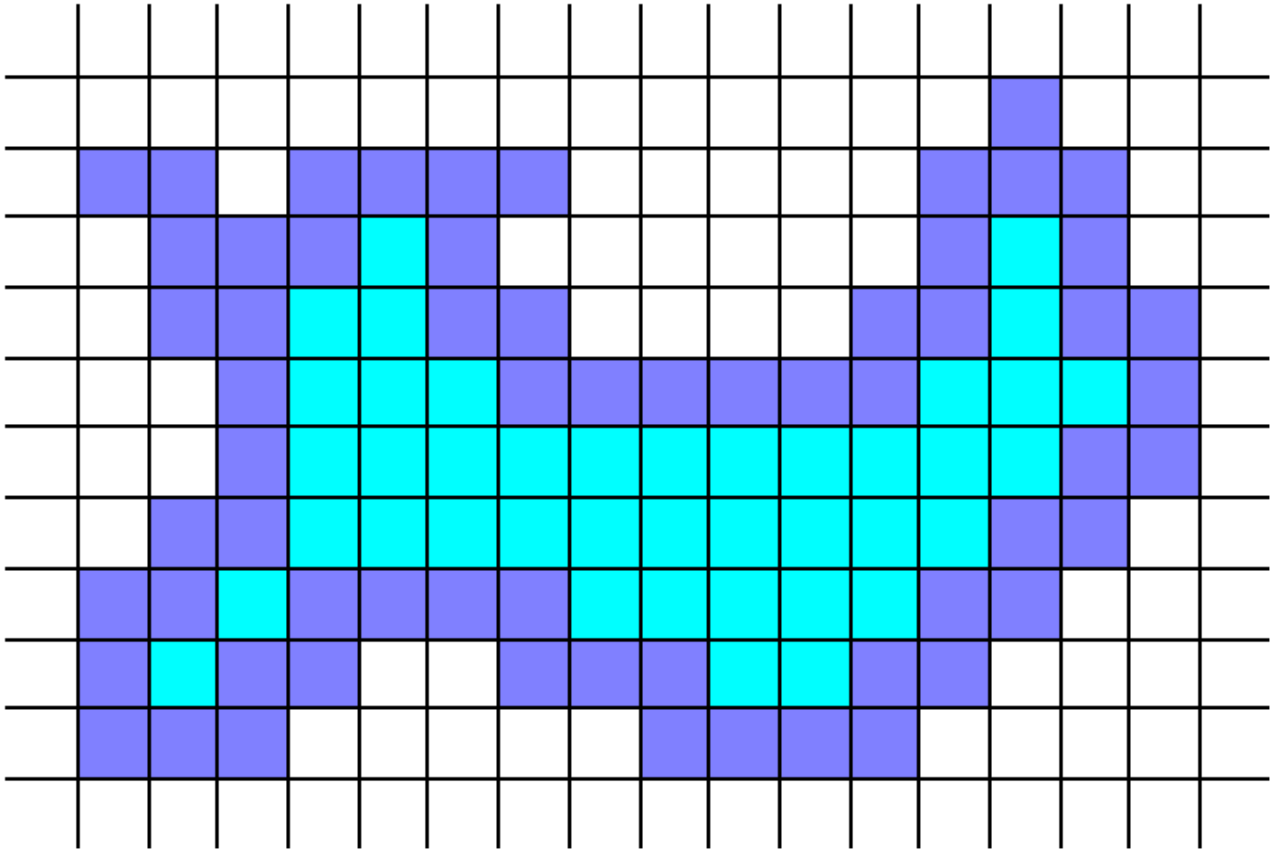

Multivalued combinatorial maps

- $\mathcal{X} \subset \mathcal{K}^d$ — a finite subfamily
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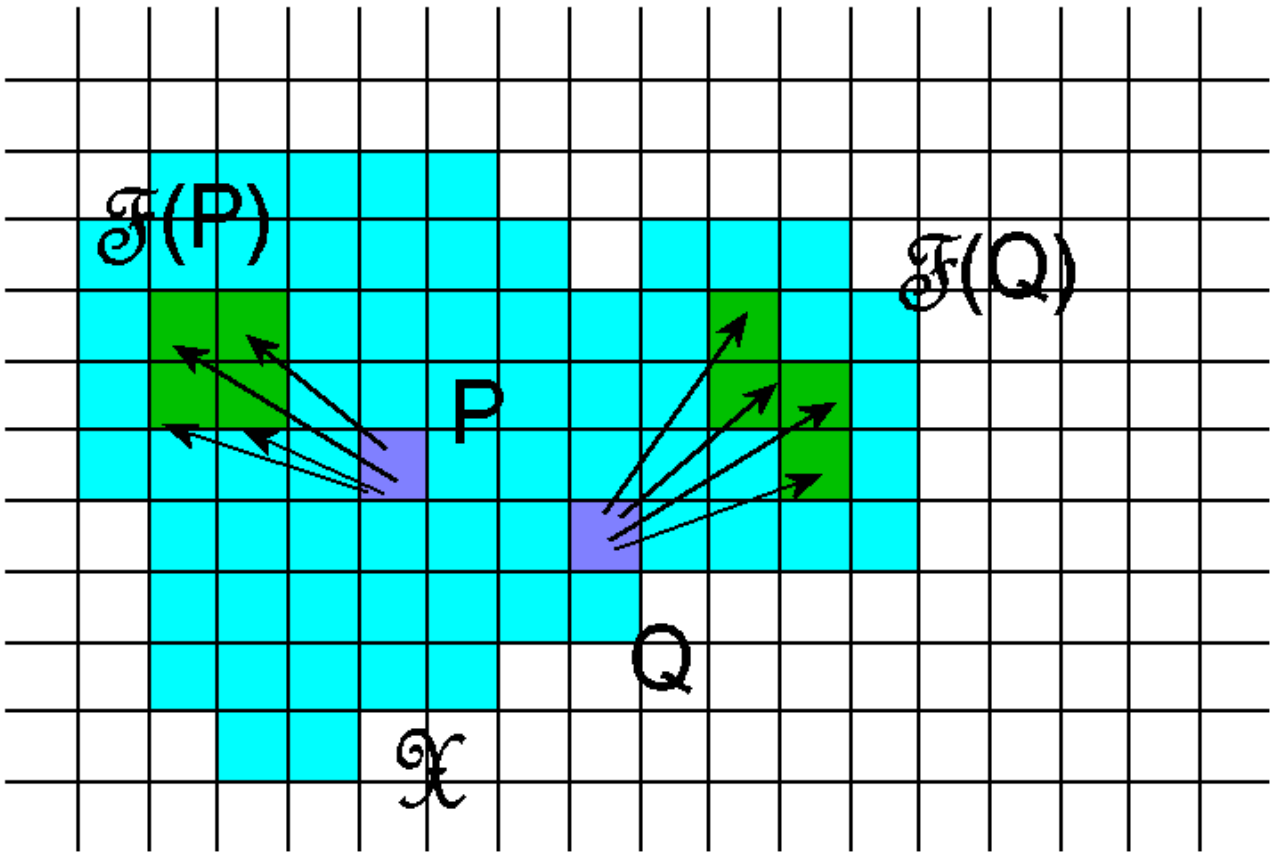
- $\mathcal{X} \subset \mathcal{K}^d$ — a finite subfamily
- $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ — a multivalued combinatorial map
- The associated digraph has $\mathcal{X}$ as the set of vertices and an edge from P to Q iff $Q \in \mathcal{F}(P)$.

Combinatorial boundary and interior



Combinatorial enclosures

- A combinatorial multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$ if for every $Q \in \mathcal{X}$

$$o_d(f(Q)) \subset \mathcal{F}(Q).$$

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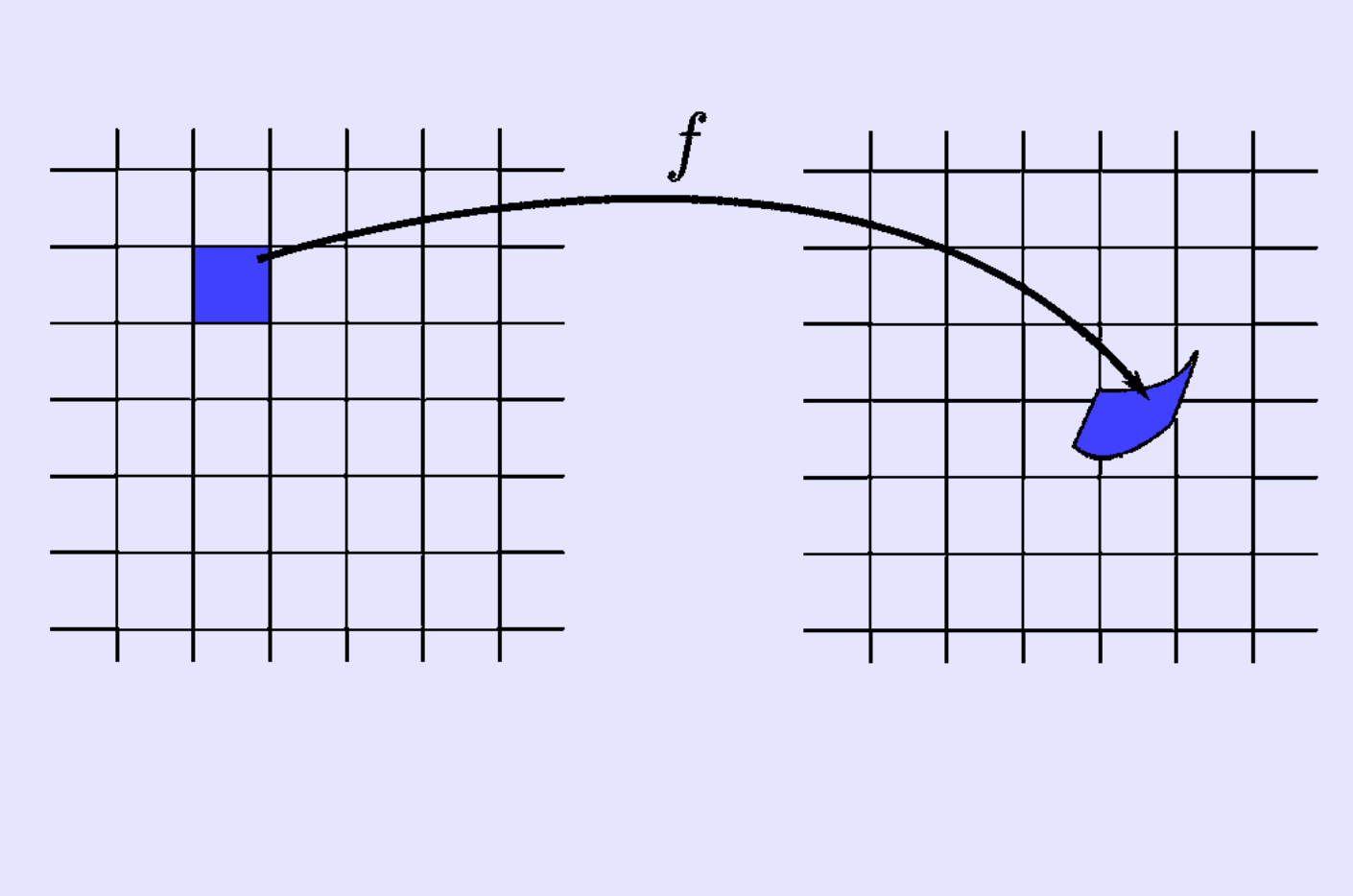
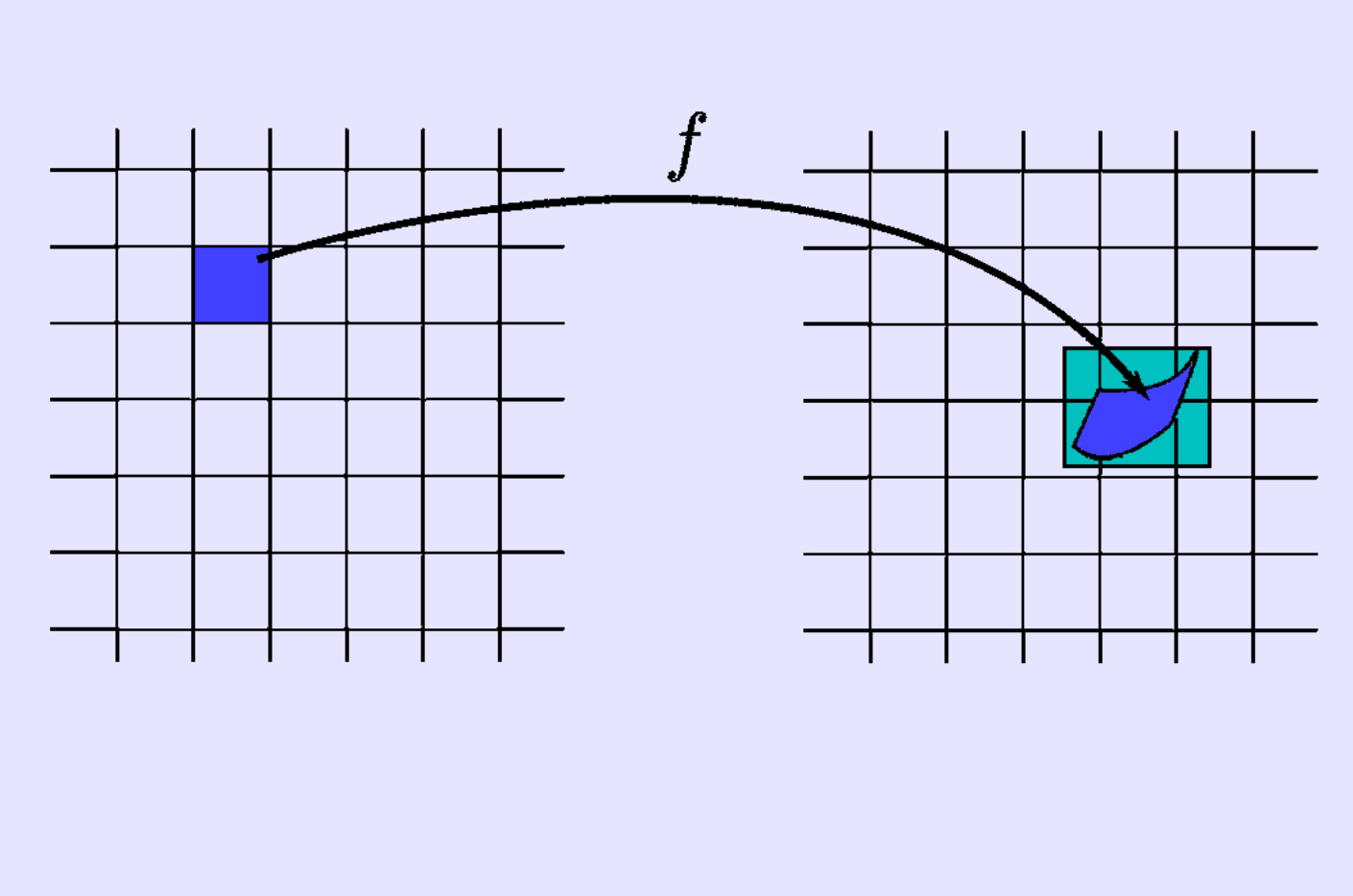
- In this case we say that f is a selector of $\mathcal{F}$.
- If $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$, then for every $Q \in \mathcal{X}$
$$f(Q) \subset \operatorname{int} |\mathcal{F}(Q)|.$$

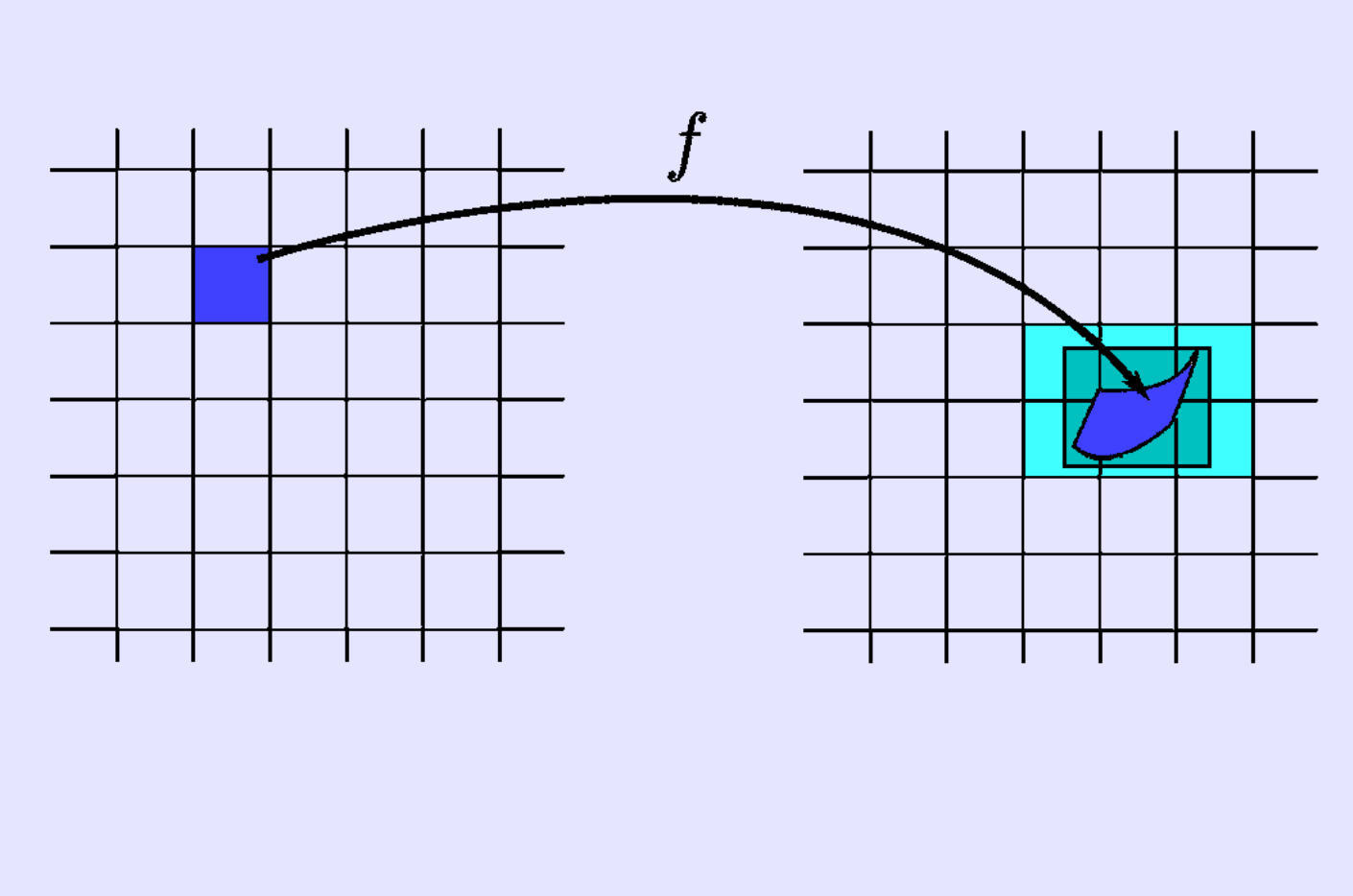
Combinatorial enclosures



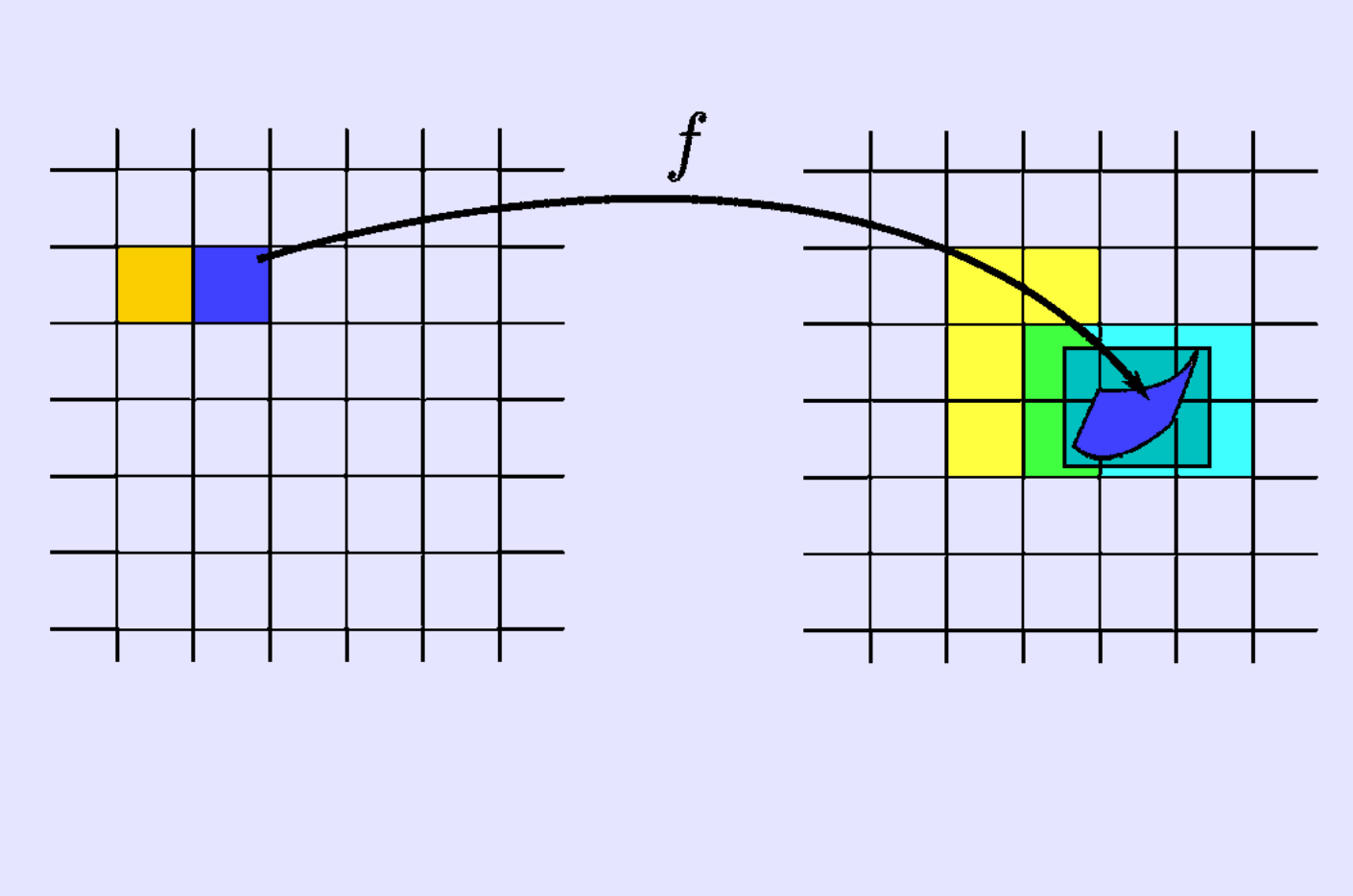
Combinatorial enclosures



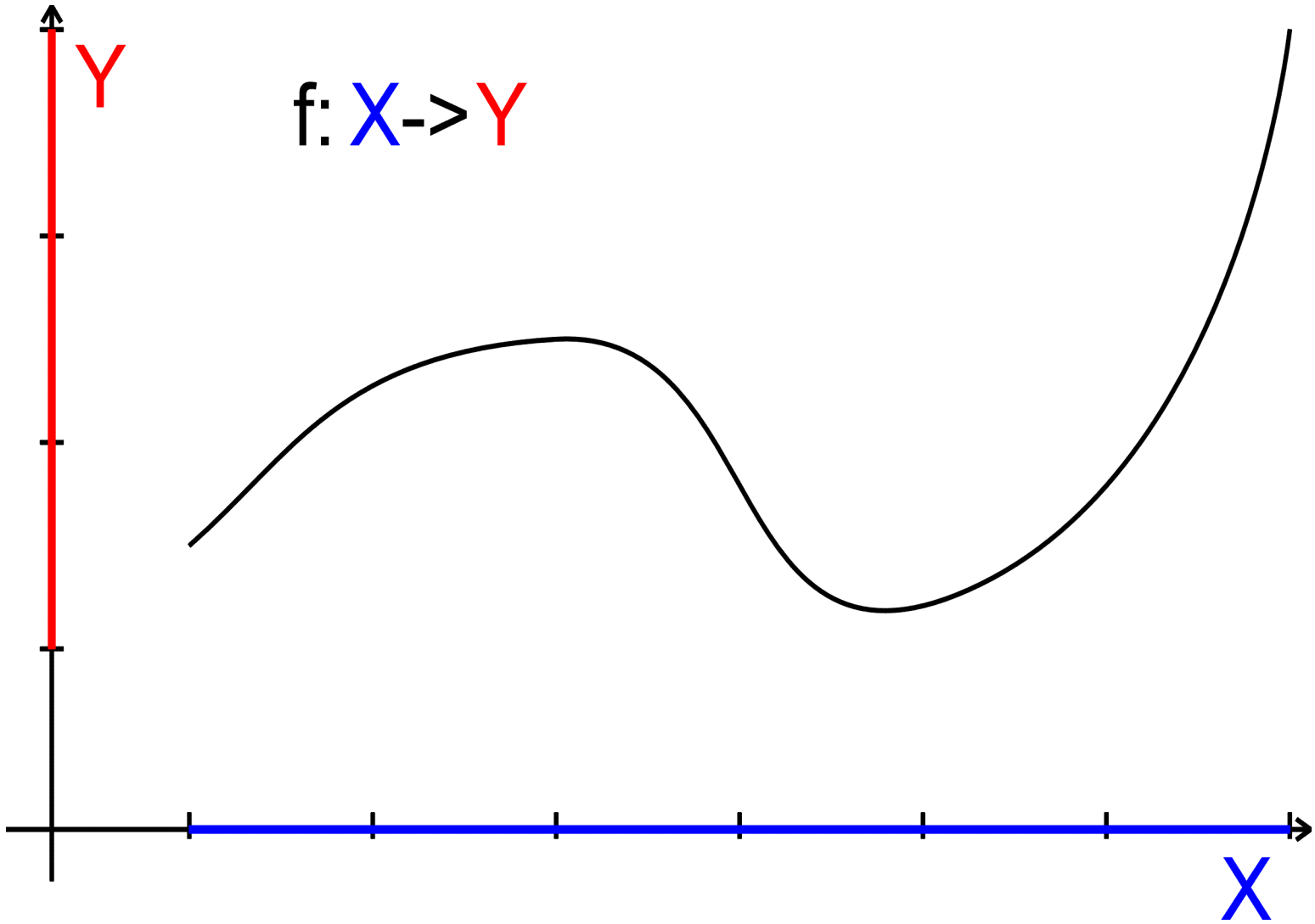
Combinatorial enclosures



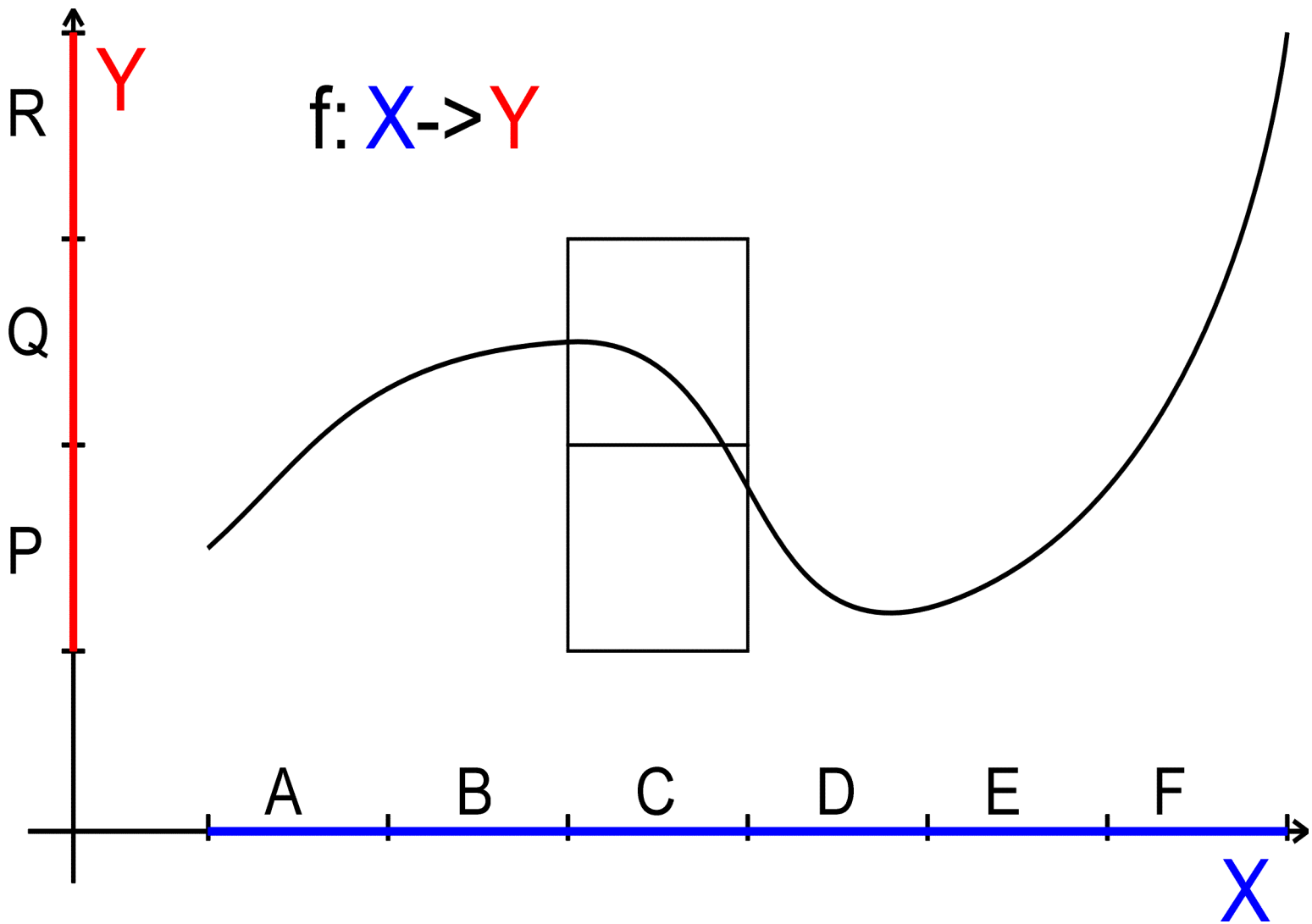
Combinatorial enclosures



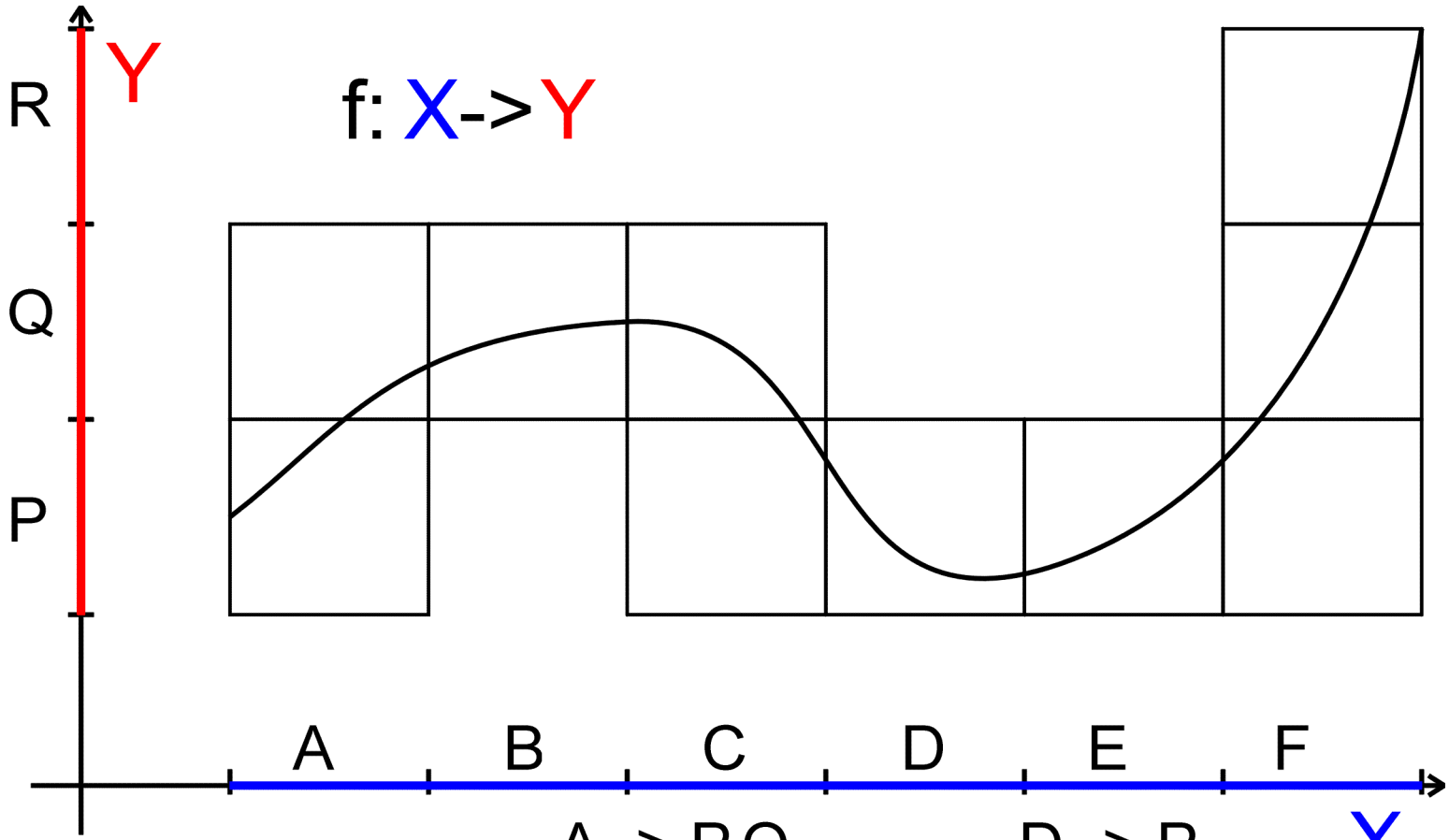
Graph of a continuous map f



Estimates of values on the grid of cubes



Multivalued representation $\mathcal{F}$



$\mathcal{F}$:

A -> P,Q
B -> Q
C -> P,Q
D -> P
E -> P
F -> P,Q,R

Combinatorial solutions

- Let I be an interval in $\mathbb{Z}$ containing 0.
- A solution through $Q \in \mathcal{K}$ under $\mathcal{F}$ is a function $\Gamma : I \to \mathcal{K}$ satisfying the following two properties:

(1) $\Gamma(0) = Q$,
(2) $\Gamma(n+1) \in \mathcal{F}(\Gamma(n))$ for all n such that $n, n+1 \in I$.

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- In the language of the associated digraph a solution is just a path in the digraph

Combinatorial invariant parts

Assume $\mathcal{N} \subset \mathcal{K}$ is finite. The invariant part of $\mathcal{N}$ under $\mathcal{F}$ is

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) := \{\, Q \in \mathcal{N} \mid \text{there exists a full solution } \Gamma : \mathbb{Z} \to \mathcal{N} \,\}.$$

Combinatorial invariant parts

Assume $\mathcal{N} \subset \mathcal{K}$ is finite. The invariant part of $\mathcal{N}$ under $\mathcal{F}$ is

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) := \{\, Q \in \mathcal{N} \mid \text{there exists a full solution } \Gamma : \mathbb{Z} \to \mathcal{N} \,\}.$$

The positively invariant part and the negatively invariant part of $\mathcal{N}$ under $\mathcal{F}$ are defined respectively by

$$\begin{aligned} \mathrm{Inv}^{+}(\mathcal{N}, \mathcal{F}) &:= \{\, Q \in \mathcal{N} \mid \text{there exists a solution } \Gamma : \mathbb{Z}^{+} \to \mathcal{N} \,\} \\ \mathrm{Inv}^{-}(\mathcal{N}, \mathcal{F}) &:= \{\, Q \in \mathcal{N} \mid \text{there exists a solution } \Gamma : \mathbb{Z}^{-} \to \mathcal{N} \,\} \end{aligned}$$

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We have the following obvious formula

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) = \mathrm{Inv}^-(\mathcal{N}, \mathcal{F}) \cap \mathrm{Inv}^+(\mathcal{N}, \mathcal{F}).$$

Algorithmizable formulae for invariant parts

Let $\mathcal{F}_{\mathcal{N}} : \mathcal{N} \rightrightarrows \mathcal{N}$ denote the map given by

$$\mathcal{F}_{\mathcal{N}}(Q) := \mathcal{F}(Q) \cap \mathcal{N}.$$

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There exists an integer n such that

$$\begin{aligned} \mathrm{Inv}^{+}(\mathcal{N}, \mathcal{F}) &= \bigcap_{i=0}^{n} \mathcal{F}_{\mathcal{N}}^{i}(\mathcal{N}) \\ \mathrm{Inv}^{-}(\mathcal{N}, \mathcal{F}) &= \bigcap_{i=0}^{n} \mathcal{F}_{\mathcal{N}}^{-i}(\mathcal{N}) \end{aligned}$$

Combinatorial Index Pairs.

A finite subset $\mathcal{N}$ of $\mathcal{K}_d$ is an isolating neighborhood for $\mathcal{F}$ if

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathrm{int}\,\mathcal{N}.$$

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- (positive relative invariance)

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$$\mathcal{F}(\mathcal{P}_1) \cap \mathrm{bd}\,\mathcal{N} \subset \mathcal{P}_2$$

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- (positive relative invariance)
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- (exit set)
$$\mathcal{F}(\mathcal{P}_1) \cap \mathrm{bd}\,\mathcal{N} \subset \mathcal{P}_2$$
- (isolation)
$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{P}_1 \setminus \mathcal{P}_2$$

Index Pairs from Combinatorial Index Pairs.

Theorem. (A. Szymczak 1997, MM 1996,2006)
Assume $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$ and $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$. Then for any selector f of $\mathcal{F}$ the set $|\mathcal{N}|$ is an isolating neighborhood for f and $(|\mathcal{P}_1|, |\mathcal{P}_2|)$ is a index pair for f.

Construction of index quadruples

Theorem. (MM,2005)
Assume $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$. Let

$$\begin{aligned}\mathcal{P}_1 &:= \mathrm{Inv}^-(\mathcal{N},\mathcal{F}),\\ \mathcal{P}_2 &:= \mathrm{Inv}^-(\mathcal{N},\mathcal{F}) \setminus \mathrm{Inv}^+(\mathcal{N},\mathcal{F}).\end{aligned}$$

Then $(\mathcal{P}_1,\mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$ and

$$|\mathcal{P}_1| \setminus |\mathcal{P}_2| \subset \mathrm{int}\,|\mathcal{N}|.$$

Moreover, if

$$\begin{aligned}\bar{\mathcal{P}}_1 &:= \mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_1),\\ \bar{\mathcal{P}}_2 &:= \mathcal{P}_2 \cup (\mathcal{F}(\mathcal{P}_1) \setminus \mathcal{P}_1),\end{aligned}$$

then for any selector f of $\mathcal{F}$ the quadruple $(|\mathcal{P}_1|,|\mathcal{P}_1|,|\bar{\mathcal{P}}_1|,|\bar{\mathcal{P}}_2|)$ is an index quadruple.

Positive invariant part algorithm

```
function positiveInvariantPart(set N, combinatorialMap F)
F := restrictedMap(F, N);
S := C := N;
repeat
  S' = S;
  C := evaluate(F, C);
  S := S ∩ C;
until (S = S');
return S;
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Proposition. Assume the algorithm is called with a collection of cubes $\mathcal{N}$ and a combinatorial multivalued map $\mathcal{F}$ on input. Then it always stops and returns the positive invariant part of $\mathcal{F}$ in $\mathcal{N}$.

Combinatorial Index Pair Algorithm

function $\texttt{combinatorialIndexPair}(\texttt{set N}, \texttt{combinatorialMap F})$
$\texttt{S}^+ := \texttt{positiveInvariantPart}(\texttt{N}, \texttt{F})$;
$\texttt{Finv} := \texttt{evaluateInverse}(\texttt{F})$;
$\texttt{S}^- := \texttt{positiveInvariantPart}(\texttt{N}, \texttt{Finv})$;
if $\texttt{S}^- \cap \texttt{S}^+ \subset \texttt{int}(\texttt{N})$ **then**
  $\texttt{P}_1 := \texttt{S}^-$;
  $\texttt{P}_2 := \texttt{S}^- \setminus \texttt{S}^+$;
  $\bar{\texttt{P}}_1 := \texttt{P}_1 \cup \texttt{F}(\texttt{P}_1)$;
  $\bar{\texttt{P}}_2 := \texttt{P}_2 \cup \texttt{F}(\texttt{P}_1) \setminus \texttt{P}_1$;
  return $(\texttt{P}_1, \texttt{P}_2, \bar{\texttt{P}}_1, \bar{\texttt{P}}_2)$;
else
  return $"Failure"$;
endif;

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  return $(\texttt{P}_1, \texttt{P}_2, \bar{\texttt{P}}_1, \bar{\texttt{P}}_2)$;
else
  return $"Failure"$;
endif;

Theorem. Assume the algorithm is called with a collection of cubes $\mathcal{N}$ and a combinatorial enclosure of f on input. If it does not fail, then it returns representations of an index quadruple of f.

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