Discrete Morse Theory

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References

References:

- Milnor, Morse theory, 1963
- R. Forman, Morse Theory for Cell Complexes Advances in Math., vol. 134, pp. 90-145, 1998
- R. Forman, User's guide to discrete Morse theory,
- Kozlov, Combinatorial algebraic topology, chapter 11

What is discrete Morse theory?

A combinatorial construction on simplicial complexes (or more generally regular cell complexes) which

- is a convenient tool for analyzing the topology of the complex
- mimicks smooth Morse theory,
- extends it to general complexes (not necessarily triangulated manifolds),
- can be easily implemented in the form of algorithms.

A smooth Morse function

Marston Morse, 1920's, reference: Milnor, Morse theory

M a smooth manifold (without boundary), $f : M \to \mathbb{R}$ smooth

A point $a \in M$ is a *critical point* of f if $Df(a) = 0$, that is, in a local coordinate system, all partial derivatives vanish at a.

A critical point is *nondegenerate*, if the matrix of second order derivatives $H(a)$ has maximal rank. The *index* of a critical point a is the number of negative eigenvalues of $H(a)$, i.e. the number of independent directions in which the function values decrease.

f is a *Morse function* if it has only nondegenerate critical points.

Morse functions are generic.

Digression: CW complexes

A *d-cell* σ is a topological space homeomorphic to the closed unit ball $B^d \subset \mathbb{R}^d$. Its *boundary* $\partial\sigma$ is the part corresponding to $S^{d-1} \subset B^d$.

Attaching a cell σ to a topological space X along an *attaching map* $f : \partial\sigma \to X$ produces the space

$$X \cup_f \sigma = X \amalg \sigma /_{s \sim f(s), s \in \partial\sigma}$$

Attaching cells along homotopic attaching maps produces homotopy equivalent spaces.

CW complexes

A *CW complex* is a finite nested sequence

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where X_i is obtained by attaching a cell to X_{i-1}.

The order of attaching can be rearranged so that the dimension of the cells increases.

The *m-skeleton* $X^{(m)}$ is the union of all cells of dimension $d \leq m$.

The cellular homology of a CW complex is computed from a chain complex generated by the cells.

Sublevel sets

The set $M_a = \{x \in M \mid f(x) \leq a\}$ is the *sublevel set* of f at $a \in \mathbb{R}$.

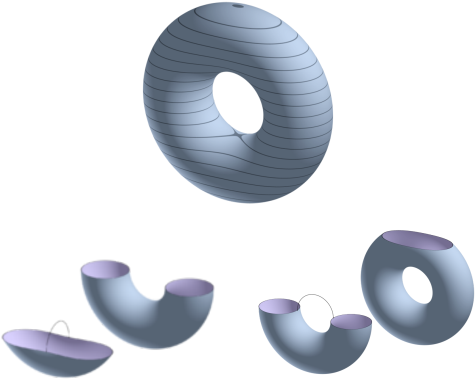
Assume that $a < b$ and $f^{-1}([a,b])$ is compact.

- If $f^{-1}([a,b])$ contains no critical points, then M_a is a deformation retract of M_b.
- If $f^{-1}([a,b])$ contains only one critical point p of index i, $a < f(p) < b$, then M_b has the homotopy type of M_a with one cell of dimension i attached.

The critical points of a smooth Morse function on M determine the homotopy type of M:
M has the homotopy type of a CW complex with one cell of dimension m for each critical point of index m.

The usual example

M upright torus, $f : M \to \mathbb{R}$ height function:



from http://en.wikipedia.org/wiki/Morse_theory

Morse homology

Morse complex

$$C = \cdots \to C_i \to C_{i-1} \to \cdots\cdots \to C_1 \to C_0 \to 0,$$

C_i free group generated by the critical points of f of index i,
boundary maps $\partial_i : C_i \to C_{i-1}$ a bit complicated ...

Morse homology is isomorphic to the singular homology:
$H_*(C) \cong H_*(M, \mathbb{Z})$.

Morse inequalities: if c_d is the number of critical points of index d and b_d is the d-th Betti number, then for all d

$$c_d \geq b_d, \quad \chi(M) = c_0 - c_1 + c_2 - \dots,$$

$$c_d - c_{d-1} + \cdots + (-1)^d c_0 \geq b_d - b_{d-1} + \cdots + (-1)^d b_0.$$

Applications

...impressive, here are just a few, topologically oriented

- Geodesics on Riemannian manifolds
- Bott periodicity theorem: homotopy groups of classical Lie groups are periodic, as a consequence K-theory is periodic
- Smale's h-cobordism theorem leading to a proof of the Poincaré conjecture in dimension $n \geq 5$
- many generalizations leading to further impressive results...

Extensions to the PL and discrete settings

(The list is definitely incomplete ...)

- Banchoff, Morse theory of PL functions on polyhedral manifolds 1967
- Goresky and MacPherson, Stratified Morse theory, 1988
- Karron and Cox, Digital Morse theory, 1994, applications to isosurface reconstruction
- Edelsbrunner, Harer, Zomorodian: a classification of PL critical points leading to PL Morse-Smale complexes for 2-manifolds, 2006
- Bestwina, PL Morse theory, 2008
- ...

Discrete Morse functions

M a regular CW complex (for example, a simplicial or cubical complex)

A *discrete Morse function* F on M is a labelling of the cells of M which associates a value $F(\sigma)$ to each cell $\sigma \in M$ such that

- F increases with dimension, excepts possibly in one direction,
- that is, for every $\sigma^k \in M$
  - $F(\tau^{k-1}) \geq F(\sigma^k)$ for at most one face $\tau < \sigma$,
  - $F(\tau^{k+1}) \leq F(\sigma^k)$ for at most one coface $\tau > \sigma$,
  - at most one of these two possibilities can happen.

Discrete vector field of F

A discrete Morse function F on a cell complex M defines a *partial pairing* on the set of cells which we call the *discrete vector field* of F.

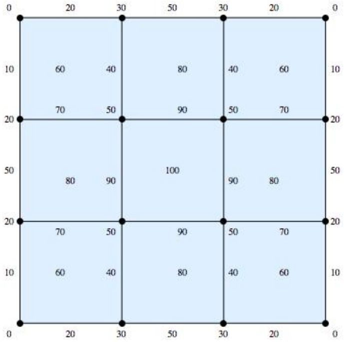
$$V = \{(\tau, \sigma), \tau < \sigma, F(\tau) \geq F(\sigma)\}.$$

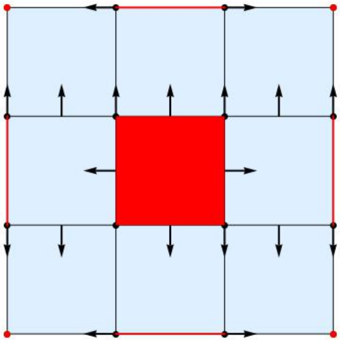
V contains all *regular cells*.

All cells not in V are *critical*.

V is conveniently denoted by arrows pointing in the direction of function decent from lower to higher dimensional cells.

Example: Discrete Morse function on the torus





Digression: elementary collapses

Let $\tau < \sigma$, and assume that τ is not the face of any other cell.

An *elementary collaps* is obtained by pushing the free face τ of σ together with the whole cell onto the remaining faces.

The resulting space has the same homotopy type.

A pair of regular cells (τ, σ) with τ a free face corresponds to an elementary collaps.

Sublevel complexes

The sublevel complex at the value c consists of all cells with value less than c together with their faces:

$$M_c = \bigcup_{F(\alpha)\leq c} \bigcup_{\beta\leq\alpha} \beta$$

If $F^{-1}((a,b])$ contains no critical cells, M_b collapses to M_a.

Proof: M_b is obtained by adding a cell σ and its pair $\tau < \sigma$ in V which must be a free face.

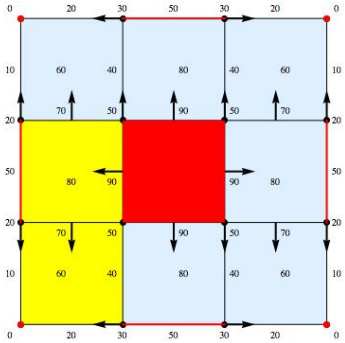
If α is the unique critical cell with $F(\alpha) \in (a, b]$ then M_b is homotopy equivalent to M_a with a cell of dimension $\dim \alpha$ attached.

Proof: a critical cell has its boundary in a previous sublevel complex, adding the critical cell corresponds to gluing the cell onto this subcomplex along the boundary.

M has the homotopy type of a CW complex with one cell of dimension m for each critical cell of dimension m.

V-paths

A *V-path* is a sequence $(\tau_1, \sigma_1), (\tau_2, \sigma_2), \ldots, (\tau_n, \sigma_m)$, where $(\tau_i, \sigma_i) \in V$ and $\tau_{i+1} < \sigma_i$ and $\sigma_i \neq \sigma_j$ for all $i \neq j$.



Along a V-path function values *descend*. Clearly, a V-path can not form cycles.

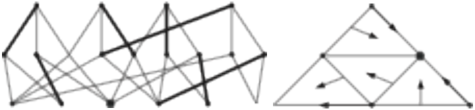
A combinatorial approach

A discrete gradient vector field can be represented as a *partial matching* in the Hasse diagram of the face poset of M.

Originally arrows in the Hasse diagram point from cells to their faces, *reverse* all arrows belonging to the partial matching.

A discrete gradient vector field is an *acyclic* partial matching, that is, after reversing the arrows there are no directed cycles.

A V-path corresponds to a directed path in the modified Hasse diagram which alternates between two levels: one segment belongs to the face poset and one to the matching.



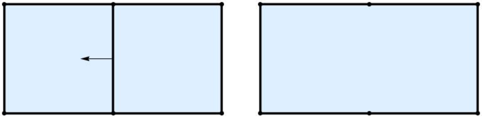
Characterization of discrete gradient vector fields

A discrete vector field V is the gradient field of a discrete Morse function if and only if the corresponding partial matching is acyclic, that is, no V-path forms a cycle.

The proof amounts to assigning values along V-paths: we start by assigning the value d to every critical cell of dimension d, and continue assigning decreasing values along V-paths, alternating between the levels $d-1$ and d, from the interval $(d-1, d)$ and where two V-paths meet (causing a conflict), the lower value wins.

Internal collapses

A matching with a single pair (τ, σ) is acyclic, such a pair can be removed in a suitable way without affecting the homotopy type of the complex.



How do we formalize this?

A *poset map with small fibers* between posets P and Q is a map $\varphi\colon P \to Q$ such that each fiber $\varphi^{-1}(q)$ is either empty or consists of one element or of two comparable elements.

A poset map with small fibers induces an acyclic partial matching and conversely, every acyclic matching induces a poset map with small fibers.

A poset map with small fibers $\varphi\colon P \to Q$ where Q is a chain determines an order of allowed internal collapses.

Discrete Morse homology

The discrete Morse complex arising from a discrete Morse function on the cell complex M

$$\to \cdots C_i \to C_{i-1} \to \cdots \to C_1 \to C_0 \to 0,$$

where C_i is now generated by critical cells of dimension i,

$$\partial_i : C_i \to C_{i-1}$$

is computed from the V-paths starting in the boundary of a cell σ^i and ending in critical cells of dimension $i-1$.

Morse inequalities: if c_d is the number of critical d-cells and b_d is the d-th Betti number, then for all d

$$c_d \geq b_d, \quad \chi(X) = c_0 - c_1 + c_2 - \dots,$$

$$c_d - c_{d-1} + \cdots + (-1)^d c_0 \geq b_d - b_{d-1} + \cdots + (-1)^d b_0.$$

Canceling critical cells

A simple way of reducing the complexity:

Cancel two critical cells which are connected by only one V-path by reversing the arrows along this single path, or, equivalently, by reversing the arrows in the corresponding partial matching on the Hasse diagram.

This introduces no cycles, so the resulting discrete vector field in also arises from a discrete Morse function.

This reduces the complexity of the discrete Morse complex and thus the computation of the homology groups.

Cancelling critical points mimics the process of *cancelling handles* in smooth Morse theory . . .

Some obvious conclusions

- If there exists a *complete acyclic matching* on the Hasse diagram (with the empty set added as a cell of dimension -1) of M, then M collapses to a point, in particular, it is contractible.
  A V-path starting in a free face goves an order of collapses.
- If M has only two critical cells, one in dimension 0 and one in dimension d, it is homotopy equivalent to a d-sphere. If it has one critical cell of dimension 0 and n critical cells of dimension d, it is homotopy equivalent to e wedge of d-spheres.
- Less obvious: if M is a triangulated d-manifold with a discrete Morse function with exactly two critical simplices. Then it is a triangulated d-sphere.

Perfect discrete Morse functions

A discrete Morse function with the number c_d of critical cells of dimension d equal to the Betti number b_d for all d is *perfect*.

There exist regular cell complexes of dimension 2 that do not have a perfect discrete Morse function, noncollapsible contractible complexes (a famous example is Bing's house with two rooms).

On 2-dimensional complexes, starting with any discrete Morse function, an optimal (the best possible) discrete Morse function can be obtained by canceling critical cells.

In dimension 3 or more this is not true any more, there exist complexes (even triangulated manifolds) with no perfect discrete Morse function.

Benedetti, Adiprasito: all tight complexes (generalization of convex cells) in $\mathbb{R}^n$ admit a perfect discrete Morse function ...

Digression to smooth theory: The Morse-Smale complex

Thom(1940), Smale(1960's)

M a Riemannian manifold, $f : M \to \mathbb{R}$ a smooth Morse function.

- The *descending disk* A_p of a critical point p is the union of all flow lines of the gradient flow which begin in p, and the *ascending disk* D_p is the union of all flow lines which end in p
- If all ascending and descending disks intersect transversely, their intersections form a CW-complex, the *Morse-Smale complex* of f on M.
- The function is a *Morse-Smale function*, these are generic among Morse functions.
- The height function of the upright torus is not Morse-Smale...

Extensions to infinite complexes

Morse theory works for *proper functions*.

A discrete Morse function F is proper in $F^{-1}([a, b])$ consists of a finite number of cells.

So, when is a discrete vector filed the gradient field of a proper function?

Ayala, Vilches, Jerše, M: Let V be a discrete vector field on a locally finite infinite simplicial complex M with finitely many critical elements which contains no nontrivial closed V-paths and no forbidden configurations. Then V admits a proper integral.

Discrete versus smooth Morse theory

Gallais: a smooth Morse function f on a closed Riemannian manifold M can be approximated by a discrete Morse function F on a triangulation of M so that critical points of f correspond to critical cells of F of dimension d.

If, in addition, the function f is Morse-Smale, the V -paths connecting any pair of critical cells σ and τ of consecutive dimensions are in bijection with integral curves of the gradient vector field of f between the corresponding critical points.

The Morse complexes are thus equivalent.

Opposite direction: if M is a smooth manifold, and a discrete Morse function on a triangulation of M is given, is it the discrete approximation of a smooth Morse function f on M?

An example from graphs

Let K be a given subcomplex of the n-dimensional simplex S with vertices $v_1, \ldots, v_n$.

For an unknown simpex $\sigma \in S$ the point is to determine whether σ belongs to K by testing which vertices of S belong to σ. The complex K is *nonevasive* if it suffices to test less that all the $n+1$ vertices.

An algorithm for choosing vertices corresponds to a gradient vector filed in the complex K.

It can be shown that if K is nonevasive, it collapses to a point.

Applications to topological data analysis

Given function values f_i at points x_i we would like to determine properties of the function f.

- A cell decomposition of the domain:
  - in some domains (for example images and image sequences) the decomposition is given
  - a number of algorithms exists
- Extending the given function values on the vertices to a discrete Morse function:
  - it suffices to define a discrete gradient vector field which respects the given function values
  - King, Knudsen, M. a recursive algorithm which, for every vertex, extends the discrete gradient field defined on the lower link to the lower star, and optimizes by canceling at each step, optimal in dimension 2,
  - the global min is correct, the global max is a cell with the maximal value in its boundary.

Cancelling

Cancelling enables noise reduction:

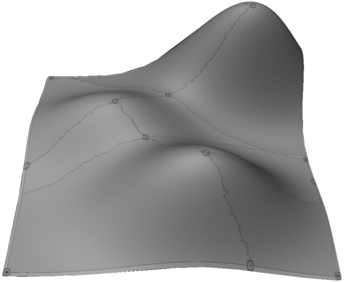
- Cancel along V-paths with values in initial and final point differing by less than some threshold.
- Use background data!
- Problem: how to control the canceling, which are the critical cells which survive?

Discrete ascending and descending regions

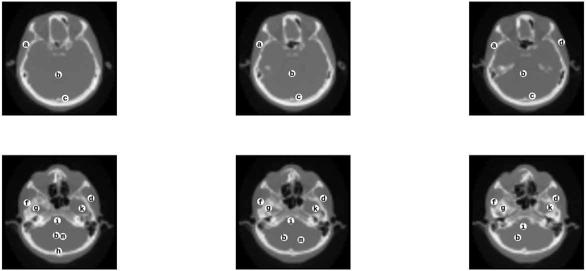
Jerše, M: an algorithm for construction:

- the descending region of a given given critical cell σ of some discrete Morse function F (or rather its discrete gradient vector field) is obtained, with some corrections, as the union of V-paths starting in σ,
- the ascending regions are the descending regions of $-f$,
- after subdivision, the descending regions are discs,
- works in any dimension, practically tested on dimensions up to 4

Example



Images



Tracing features

F_{t_i} a finite family of discrete Morse fuctions on a regular cell complex M

- *Birth-death algorithm* (King, Knudson, M.):
- for each i the strip $M \times [t_{i-1}, t_i]$ is decomposed into cells $\sigma \times [t_{i-1}, t_i]$,
- the given discrete vector field is extended from F_{t_i} on the slice $M \times \{t_i\}$ to the strip mimicking a time decreasing function,
- critical cells σ in the slice are paired with $\sigma \times [t_{i-1}, t_i]$,
- this produces a V-path connecting a critical cell in $M \times \{t_i\}$ with a critical cell in the previous slice,
- the result is a bifurtacation diagram connecting the critical points.
- *Problem:* the choice is not natural, background information should be included,